1111: Linear Algebra I

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Lecture 8

Odd and even permutations

Let σ be a permutation of n elements, written in the one-row notation. Two numbers i and j, where $1 \leqslant i < j \leqslant n$, are said to form an inversion in σ, if they are "listed in wrong order", that is j appears before i in σ.

For the permutation $1, 3, 4, 2$, there are 6 pairs (i, j) to look at: $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, $(3, 4)$. Of these, the pair $(2, 3)$ forms an inversion, and the pair $(2, 4)$ does, and other pairs do not.

A permutation is said to be *even* if its number of inversions is even, and *odd* otherwise. One of the most important properties of this division into even and odd is the following: *if we swap two numbers in a permutation a_1, ..., a_n, it makes an even permutation into an odd one, and vice versa*.

Let us first remark that this is obvious if we swap two neighbours, a_p and a_{p+1}. Indeed, this only changes whether they form an inversion or not, since their positions relative to others do not change. Now, a swap of a_p and a_q can be done by dragging a_p through a_{p+1}, ..., a_{q-1}, swapping it with a_q, and dragging a_q through a_{q-1}, ..., a_{p+1}, so altogether we do an odd number of "swapping neighbours", and changed odd to even / even to odd.

Odd and even permutations

The property that we just proved is useful for a yet another definition of even / odd permutations that refers to the two-row notation. Namely, a permutation in two-row notation is even if the total number of inversions in the top and the bottom row is even, and is odd, if the total number of inversions in the top and the bottom row is odd.

The usual problem to address is whether this definition makes sense or not: maybe it will give different answers for different two-row representations. Luckily, it is not the case: different representations are obtained from one another by re-arranging columns, and each swap of columns will change the number of inversions in the top row and in the bottom row by odd numbers, so altogether will give a change by an even number.

For example, the two different representations of $1, 3, 4, 2$, that we discussed, we observe that for $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, the total number of inversions is 2, and for $\begin{pmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 4 & 3 \end{pmatrix}$ the total number of inversions is $3 + 1 = 4$.

Determinants

We shall now define an important numeric invariant of an $n \times n$-matrix A, the *determinant* of A, denoted $\det(A)$. Informally, the determinant of A is the signed sum of all possible products of n entries of A, chosen in a way that every row and every column is represented in the product exactly once.

Formally,

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} \ .$$

Here σ runs over all permutations of n elements, and $\begin{pmatrix} i_1 & i_2 & \ldots & i_n \\ j_1 & j_2 & \ldots & j_n \end{pmatrix}$ is some two-row representation of σ. The *sign* $\operatorname{sgn}(\sigma)$ of a permutation σ is defined to be 1 if σ is even, and -1 if σ is odd.

Examples of determinants

For a 2×2-matrix $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, we have

$$\det(A)=ad-bc.$$

Indeed, there are two permutations of $1,2$: $1,2$ and $2,1$, the first one even, the second one odd.

For a 3×3-matrix $A=\begin{pmatrix} a & b & c\\ d & e & f\\ g & h & k\end{pmatrix}$, we have

$$\det(A)=aek+bfg+cdh-ceg-afk-bdk.$$

An easy way to memorise: copy two first columns of A next to it

$$\begin{pmatrix} a & b & c & a & b\\ d & e & f & d & e\\ g & h & k & g & h\end{pmatrix},$$

then multiply along the six "diagonals", and take northwest diagonals with the plus sign and northeast diagonals with the minus sign. Does **not** work for $n>3$!!!!

Examples of determinants

If matrix A is a *diagonal* matrix (with entries d_1, ..., d_n on the diagonal and zeros elsewhere), then $\det(A) = d_1 d_2 \cdots d_n$, since the only way to choose n entries in n different rows of A in a way that no zeros are chosen, is to take the diagonal entries; this corresponds to the permutation $1, 2, \ldots, n$ which is clearly even.

In particular, $\det(I_n) = 1$.

More generally, if matrix A is an *upper triangular* matrix (with entries d_1, ..., d_n on the diagonal, some entries above the diagonal, and zeros below the diagonal), then $\det(A) = d_1 d_2 \cdots d_n$, since the only way to choose n entries in n different rows and n different columns of A in a way that no zeros are chosen, is to take the diagonal entries.

Properties of determinants

Let me list some properties of determinants that are most useful in calculations.

- If three matrices A, A', and A'' have all rows except for the i-th row i in common, and the i-th row of A is equal to the sum of the i-th rows of A' and A'', then $\det(A) = \det(A') + \det(A'')$;
- if two matrices A and A' have all rows except for the i-th row in common, and the i-th row of A' is obtained from the i-th row of A by multiplying it by a scalar c, then $\det(A') = c \det(A)$;
- if two matrices A and A' have all rows except for the i-th and the j-th row in common, and A' is obtained from the A by swapping the i-th row with the j-th row, then $\det(A') = -\det(A)$;
- if two matrices A and A' have all rows except for the i-th row in common, and the i-th row of A' is obtained from the i-th row of A by adding a multiple of another row, then $\det(A') = \det(A)$;

Properties of determinants

Effectively, these properties say that

- determinants are *multilinear functions* of their rows,
- determinants behave predictably with respect to elementary row operations.

Let us give an example of how this can be used:

$$\det\begin{pmatrix}1 & 2 & 2\\ 1 & 2 & 8\\ 1 & 1 & 4\end{pmatrix} \overset{(2)-(1),(3)-(1)}{=} \det\begin{pmatrix}1 & 2 & 2\\ 0 & 0 & 6\\ 0 & -1 & 2\end{pmatrix} =$$

$$(-1)\cdot 6\cdot \det\begin{pmatrix}1 & 2 & 2\\ 0 & 0 & 1\\ 0 & 1 & -2\end{pmatrix} \overset{(2)\leftrightarrow(3)}{=} 6\det\begin{pmatrix}1 & 2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix} = 6.$$