1212: Linear Algebra II

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Lecture 12

Today we shall start proving the theorems on quadratic forms that you used in the tutorial class last week.

Theorem 1. *Let* q *be a quadratic form on a vector space* V*. There exists a basis* $f_1, \ldots, f_n$ *of* V *for which the quadratic form* q *becomes a signed sum of squares:*

$$q(x_1 f_1 + \cdots + x_n f_n) = \sum_{i=1}^{n} \varepsilon_i x_i^2,$$

where all numbers ε_i *are either* 1 *or* -1 *or* 0*.*

Proof. We shall prove the appropriate statement for symmetric bilinear forms; it is a bit more transparent that way. Suppose that b is a symmetric bilinear form. If $q(v) = b(v, v) = 0$ for all v, then we have the representation with all $\varepsilon_i = 0$. Suppose that there exists a vector v with $q(v) \neq 0$. Consider some basis $e_1, e_2, \ldots, e_n = v$ of V. We claim that we can change it into a basis $f_1, f_2, \ldots, f_n$ with $b(f_n, f_n) = \pm 1$, and $b(f_i, f_n) = 0$ for all $i = 1, \ldots, n-1$. Indeed, we can replace e_n by $f_n = \frac{1}{\sqrt{|q(e_n)|}} e_n$, and then replace e_i by $f_i = e_i - \frac{b(e_i, f_n)}{b(f_n, f_n)} f_n$. Then we may consider the linear span of $f_1, \ldots, f_{n-1}$, and proceed by induction on dimension. □

Theorem 2 (Law of inertia)**.** *In the previous theorem, the triple* (n_+, n_-, n_0)*, where* $n_\pm$ *is the number of* ε_i *equal to* ± 1*, and* n_0 *is the number of* ε_i *equal to* 0*, does not depend on the choice of the basis* $f_1, \ldots, f_n$*. This triple is often referred to as the* signature *of the quadratic form* q*.*

Proof. Suppose that we have a basis

$$e_1, \ldots, e_{n_+}, f_1, \ldots, f_{n_-}, g_1, \ldots, g_{n_0}$$

which produces a system of coordinates where q becomes a signed sum of squares with $n_+$of ε_i are equal to 1, n_- of ε_i are equal to -1, and n_0 of ε_i are equal to 0. Let us note that for the corresponding bilinear form b, the subspace spanned by $g_1, \ldots, g_{n_0}$ is its kernel, that is the space of all vectors u for which $b(u, v) = 0$ for all $v \in V$. Indeed, those conditions read $b(u, e_i) = b(u, f_j) = b(u, g_k) = 0$ for all i, j, k, which by inspection shows that u is a linear combination of g_k. Suppose now that there are two different bases

$$e_1, \ldots, e_{n_+}, f_1, \ldots, f_{n_-}, g_1, \ldots, g_{n_0}$$

and

$$e'_1, \ldots, e_{n'_+}, f'_1, \ldots, f_{n'_-}, g'_1, \ldots, g'_{n_0}$$

where q is a signed sum of squares, and $n_+ \neq n'_+$, so without loss of generality $n_+ > n'_+$. Note that this implies that $n_- < n'_-$. Consider the vectors

$$e_1, \ldots, e_{n_+}, f'_1, \ldots, f'_{n'_-}, g_1, \ldots, g_{n_0}.$$

The total number of those vectors exceeds the dimension of V, so they must be linearly dependent, that is

$$a_1 e_1 + \cdots + a_{n_+} e_{n_+} + b_1 f'_1 + \cdots + b_{n'_-} f'_{n'_-} + c_1 g_1 + \cdots + c_{n_0} g_{n_0} = 0$$

for some scalars a_i, b_j, c_k. Let us rewrite it as

$$a_1 e_1 + \cdots + a_{n_+} e_{n_+} + c_1 g_1 + \cdots + c_{n_0} g_{n_0} = -(b_1 f'_1 + \cdots + b_{n'_-} f'_{n'_-}),$$

and denote the vector to which both the left hand side and the right hand side are equal to by v. Then

$$a_1^2 + \cdots + a_{n_+}^2 = q(v) = -b_1^2 - \cdots - b_{n_-}^2,$$

which implies

$$a_1 = \cdots = a_{n_+} = b_1 = \cdots = b_{n_-} = 0,$$

and substituting it into

$$a_1 e_1 + \cdots + a_{n_+} e_{n_+} + b_1 f'_1 + \cdots + b_{n'_-} f'_{n'_-} + c_1 g_1 + \cdots + c_{n_0} g_{n_0} = 0$$

we get

$$c_1 g_1 + \cdots + c_{n_0} g_{n_0} = 0,$$

implying of course $c_1 = \cdots = c_{n_0} = 0$, which altogether shows that these vectors cannot be linearly dependent, a contradiction. □