1212: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 14

Example for the Jacobi theorem

Let us begin with an example for the Jacobi theorem from last class. Suppose that a bilinear form has the matrix $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ relative to a basis e_1, e_2, e_3.

Let us compute the determinants $\Delta_1, \Delta_2, \Delta_3$. We have $\Delta_1 = 3$, $\Delta_2 = 2$, $\Delta_3 = -7$. Therefore, the conditions of the Jacobi theorem are satisfied.

We are looking for a basis of the form

$$\begin{aligned} f_1 &= \alpha_{11}e_1, \\ f_2 &= \alpha_{12}e_1 + \alpha_{22}e_2, \\ f_3 &= \alpha_{13}e_1 + \alpha_{23}e_2 + \alpha_{33}e_3, \end{aligned}$$

imposing equations $A(e_i, f_j) = 0$ for $i < j$, and $A(e_i, f_i) = 1$ for all i. This means that

$$1 = A(e_1, f_1) = 3\alpha_{11},$$

$$\begin{aligned} 0 &= A(e_1, f_2) = 3\alpha_{12} + \alpha_{22}, \\ 1 &= A(e_2, f_2) = \alpha_{12} + \alpha_{22}, \end{aligned}$$

$$\begin{aligned} 0 &= A(e_1, f_3) = 3\alpha_{13} + \alpha_{23} - \alpha_{33}, \\ 0 &= A(e_2, f_3) = \alpha_{13} + \alpha_{23} - 2\alpha_{33}, \\ 1 &= A(e_3, f_3) = -\alpha_{13} - 2\alpha_{23} + \alpha_{33}. \end{aligned}$$

Solving these linear equations, we get $\alpha_{11} = \frac{1}{3}$, $\alpha_{12} = -\frac{1}{2}$, $\alpha_{22} = \frac{3}{2}$, $\alpha_{13} = \frac{1}{7}$, $\alpha_{23} = -\frac{5}{7}$, $\alpha_{33} = -\frac{2}{7}$, so the corresponding change of basis is

$$\begin{aligned} f_1 &= \frac{1}{3}e_1, \\ f_2 &= -\frac{1}{2}e_1 + \frac{3}{2}e_2, \\ f_3 &= \frac{1}{7}e_1 - \frac{5}{7}e_2 - \frac{2}{7}e_3. \end{aligned}$$

Sylvester's criterion

Theorem 1 (Sylvester's criterion)**.** *The given symmetric bilinear form is positive definite if and only if*

$$\Delta_k > 0 \quad \textit{for all} \quad k = 1, \ldots, n.$$

Proof. Suppose that all Δ_k are positive. Then in particular they are all non-zero, and we are in the situation of Jacobi theorem, which immediately shows that b is positive definite, since $q(v) = b(v, v)$ is represented by a sum of squares of coordinates with positive coefficients.

Suppose that b is positive definite. Let us show that it is impossible to have $\Delta_k = 0$ for some k. Assume the contrary. Then the homogeneous system of linear equations

$$\begin{cases} b(e_1, e_1)x_1 + b(e_2, e_1)x_2 + \ldots + b(e_k, e_1)x_k = 0, \\ b(e_1, e_2)x_1 + b(e_2, e_2)x_2 + \ldots + b(e_k, e_2)x_k = 0, \\ \ldots \\ b(e_1, e_k)x_1 + b(e_2, e_k)x_2 + \ldots + b(e_k, e_k)x_k = 0 \end{cases}$$

has a nontrivial solution. Let us take this solution, and consider the vector $v = x_1 e_1 + \cdots + x_k e_k$. We have

$$b(v, e_i) = b(e_1, e_i)x_1 + b(e_2, e_i)x_2 + \ldots + b(e_k, e_i)x_k = 0 \text{ for } i = 1, \ldots, k.$$

Therefore,

$$b(v, v) = x_1 b(v, e_1) + x_2 b(v, e_2) + \cdots + x_k b(v, e_k) = 0,$$

which contradicts the assumption that b is positive definite. Therefore, all the determinants Δ_k are nonzero, and then the previous theorem implies that they must be positive, or else the expansion

$$q(x_1 f_1 + \cdots + x_n f_n) = \frac{1}{\Delta_1} x_1^2 + \frac{\Delta_1}{\Delta_2} x_2^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} x_n^2$$

has a negative coefficient, and so b cannot be positive definite. □

Hermitian vector spaces

We would like to adapt the results that we proved for the case of complex numbers. However, we started with defining scalar products, and for complex numbers, the notion of a positive number does not make sense. So we have to be a bit more imaginative.

Definition 1. A vector space V over complex numbers is said to be a Hermitian vector space if it is equipped with a function (Hermitian scalar product) $V \times V \to \mathbb{C}$, $v_1, v_2 \mapsto (v_1, v_2)$ satisfying the following conditions:

- sesquilinearity: $(v_1 + v_2, v) = (v_1, v) + (v_2, v)$, $(v, v_1 + v_2) = (v, v_1) + (v, v_2)$, $(cv_1, v_2) = c(v_1, v_2)$, and $(v_1, cv_2) = \bar{c}(v_1, v_2)$,
- symmetry: $(v_1, v_2) = \overline{(v_2, v_1)}$ for all v_1, v_2 (in particular, $(v, v) \in \mathbb{R}$ for all v),
- positivity: $(v, v) \geqslant 0$ for all v, and $(v, v) = 0$ only for $v = 0$.