1212: Linear Algebra II

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Lecture 17

For the rest of the semester, we shall focus on general theory of linear transformations in complex vector spaces. The main question we shall address is how to find a system of coordinates where the matrix of the given linear transformation φ has a particularly simple shape (for example, for purposes of computing φ^k, but for other purposes too). Before we deal with this problem in full generality, let us consider two examples which show various subtleties of our problem.

Case $\varphi^2 = \varphi$

In this section we are dealing with a special case of linear transformations, those satisfying $\varphi^2 = \varphi$.

Lemma 1. *If* $\varphi^2 = \varphi$, *then* $\mathrm{Im}(\varphi) \cap \ker(\varphi) = \{0\}$.

Proof. Indeed, if $v \in \mathrm{Im}(\varphi) \cap \ker(\varphi)$, then $v = \varphi(w)$ for some w, and $0 = \varphi(w) = \varphi(\varphi(w)) = \varphi^2(w) = \varphi(w) = v$. □

Not that from this proof it is clear that if $v \in \mathrm{Im}(\varphi)$, then $\varphi(v) = v$.

Lemma 2. *If* $\varphi^2 = \varphi$, *then* $V = \mathrm{Im}(\varphi) \oplus \ker(\varphi)$.

Proof. Indeed,

$$\dim(\mathrm{Im}(\varphi) + \ker(\varphi)) = \dim \mathrm{Im}(\varphi + \dim\ker(\varphi) - \dim(\mathrm{Im}(\varphi) \cap \ker(\varphi)) = \mathrm{rk}(\varphi) + \mathrm{null}(\varphi) = \dim(V),$$

so the sum is a subspace of V of dimension equal to the dimension of V, that is V itself. Also, we already checked that the intersection is equal to 0, so the sum is direct. □

Consequently, if we take a basis of $\ker(\varphi)$, and a basis of $\mathrm{Im}(\varphi)$, and join them together, we get a basis of V relative to which the matrix of φ is $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

Case $\varphi^2 = 0$

However nice the approach from the previous section seems, sometimes it does not work that well. Though we always have

$$\dim \mathrm{Im}(\varphi) + \dim \ker(\varphi) = \dim(V),$$

the sum of these subspaces is not always direct, as the following example shows. If we know that $\varphi^2 = 0$, that is $\varphi(\varphi(v)) = 0$ for every $v \in V$, that implies $\mathrm{Im}(\varphi) \subset \ker(\varphi)$, so $\mathrm{Im}(\varphi) + \ker(\varphi) = \ker(\varphi)$. Let us discuss a way to handle this case, it will be very informative for our future results. We begin with a general definition which will be useful for packaging various constructions we shall use.

Definition 1. Let U be a subspace of V.

- We say that vectors $e_1, \ldots, e_k$ are linearly independent relative to U if $c_1e_1 + \cdots + c_ke_k \in U$ implies $c_1 = \cdots = c_k = 0$.

- We say that vectors $e_1, \ldots, e_k$ form a spanning set relative to U if $\mathrm{span}(e_1, \ldots, e_k) + U = V$.
- We say that vectors $e_1, \ldots, e_k$ form a basis relative to U if they are linearly independent relative to U and form a spanning set relative to U. Alternatively, we can say that the sum of the subspaces $\mathrm{span}(e_1, \ldots, e_k)$ and U is direct and is equal to V.

Note that the usual notion of a linearly independent set, a spanning set, and a basis are obtained in the case $U = \{0\}$.

Now, let us pick a basis $e_1, \ldots, e_k$ of V relative to $\ker(\varphi)$. Note that $\varphi(e_1), \ldots, \varphi(e_k) \in \mathrm{Im}(\varphi) \subset \ker(\varphi)$. Let us pick a basis $f_1, \ldots, f_l$ of $\ker(\varphi)$ relative to $\mathrm{span}(\varphi(e_1), \ldots, \varphi(e_k))$. Let us show that the vectors

$$e_1, \ldots, e_k, \varphi(e_1), \ldots, \varphi(e_k), f_1, \ldots, f_l$$

are linearly independent (and hence form a basis of V). Suppose that

$$a_1 e_1 + \cdots + a_k e_k + b_1 \varphi(e_1) + \cdots + b_k \varphi(e_k) + c_1 f_1 + \cdots + c_l f_l = 0,$$

so that

$$a_1 e_1 + \cdots + a_k e_k = -b_1 \varphi(e_1) - \cdots - b_k \varphi(e_k) - c_1 f_1 - \cdots - c_l f_l.$$

The right hand side belongs to $\ker(\varphi)$, so since $e_1, \ldots, e_k$ is a relative basis, we conclude that $a_1, \ldots, a_k$ are all equal to zero. Therefore,

$$b_1 \varphi(e_1) + \cdots + b_k \varphi(e_k) + c_1 f_1 + \cdots + c_l f_l = 0,$$

so since $f_1, \ldots, f_l$ is a relative basis, we conclude that $c_1, \ldots, c_l$ are all equal to zero. Therefore,

$$\varphi(b_1 e_1 + \cdots + b_k e_k) = b_1 \varphi(e_1) + \cdots + b_k \varphi(e_k) = 0,$$

so $b_1 e_1 + \cdots + b_k e_k \in \ker(\varphi)$, and hence $b_1, \ldots, b_k$ are all equal to zero.

Reordering this basis, we obtain a basis

$$e_1, \varphi(e_1), \ldots, e_k, \varphi(e_k), f_1, \ldots, f_l,$$

relative to which the matrix of φ has a block diagonal form with k blocks $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ on the diagonal, and all the other entries equal to zero.