1212: Linear Algebra II

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Lecture 19

Suppose that φ is a linear transformation of a vector space V for which $\varphi^k = 0$ (such operators are called nilpotent). We shall adapt the argument that we had for $k = 2$ to this general case.

Let us put, for each p, $N_p = \ker(\varphi^p)$. Let us assume that k is actually the smallest power of φ that vanishes, so that $\varphi^{k-1} \neq 0$. Of course, we have $N_k = N_{k+1} = N_{k+2} = \ldots = V$.

We shall now construct a basis of V of a very particular form. It will be constructed in k steps. First, we find a basis of $V = N_k$ relative to N_{k-1}. Let $e_1, \ldots, e_s$ be vectors of this basis.

The following result is proved in the same way as last week:

Lemma 1. *The vectors* $e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s)$ *are linearly independent relative to* N_{k-2}.

Proof. Indeed, assume that

$$c_1e_1 + \ldots + c_se_s + d_1\varphi(e_1) + \ldots + d_s\varphi(e_s) \in N_{k-2}.$$

Since $e_i \in N_k$, we have $\varphi(e_i) \in N_{k-1}$, so

$$c_1e_1 + \ldots + c_se_s \in -d_1\varphi(e_1) - \ldots - d_s\varphi(e_s) + N_{k-2} \subset N_{k-1},$$

which means that $c_1 = \ldots = c_s = 0$. Thus,

$$\varphi(d_1e_1 + \ldots + d_se_s) = d_1\varphi(e_1) + \ldots + d_s\varphi(e_s) \in N_{k-2},$$

so $d_1e_1 + \ldots + d_se_s \in N_{k-1}$, and we deduce that $d_1 = \ldots = d_s = 0$, thus the lemma follows. □

Now we find vectors $f_1, \ldots, f_t$ which form a basis of N_{k-1} relative to $\mathrm{span}(\varphi(e_1), \ldots, \varphi(e_s)) \oplus N_{k-2}$. Absolutely analogously one can prove

Lemma 2. *The vectors* $e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s), \varphi^2(e_1), \ldots, \varphi^2(e_s), f_1, \ldots, f_t, \varphi(f_1), \ldots, \varphi(f_t)$ *are linearly independent relative to* N_{k-3}.

We continue that extension process until we end up with a basis of V of the following form:

$$\begin{gathered} e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s), \varphi^2(e_1), \ldots, \varphi^{k-1}(e_1), \ldots, \varphi^{k-1}(e_s), \\ f_1, \ldots, f_t, \varphi(f_1), \ldots, \varphi^{k-2}(f_1), \ldots, \varphi^{k-2}(f_t), \\ \ldots, \\ g_1, \ldots, g_u, \end{gathered}$$

where the first line contains several "threads" $e_i, \varphi(e_i), \ldots, \varphi^{k-1}(e_i)$ of length k, the second line — several threads of length $k-1$, ..., the last line — several threads of length 1, that is several vectors from N_1.

Let us rearrange the basis vectors so that vectors forming a thread are all next to each other:

$$\begin{gathered} e_1, \varphi(e_1), \ldots, \varphi^{k-1}(e_1), \ldots, e_s, \varphi(e_s), \ldots, \varphi^{k-1}(e_s), \\ f_1, \varphi(f_1), \ldots, \varphi^{k-2}(f_1), \ldots, f_t, \varphi(f_t), \ldots, \varphi^{k-2}(f_t), \\ \ldots, \\ g_1, \ldots, g_u. \end{gathered}$$

Relative to that basis, the linear transformation φ has the matrix made of *Jordan blocks*

$$J_l = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

one block J_l for each thread of length l.

Example 1. $V = \mathbb{R}^3$, φ is multiplication by the matrix $A = \begin{pmatrix} 21 & -7 & 8 \\ 60 & -20 & 23 \\ -3 & 1 & -1 \end{pmatrix}$. In this case, φ^2 is multiplication by the matrix $\begin{pmatrix} -3 & 1 & -1 \\ -9 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$, $\varphi^3 = 0$, $\operatorname{rk}\varphi = 2$, $\operatorname{rk}\varphi^2 = 1$, $\operatorname{rk}\varphi^k = 0$ for $k \geqslant 3$, $\operatorname{null}(\varphi) = 1$, $\operatorname{null}(\varphi^2) = 2$, $\operatorname{null}(\varphi^k) = 3$ for $k \geqslant 3$. We have a sequence of subspaces $V = \operatorname{Ker}\varphi^3 \supset \operatorname{Ker}\varphi^2 \supset \operatorname{Ker}\varphi \supset \{0\}$. The first one relative to the second one is one-dimensional ($\dim\operatorname{Ker}\varphi^3 - \dim\operatorname{Ker}\varphi^2 = 1$). We have $\operatorname{Ker}(\varphi^2) = \{\begin{pmatrix} \frac{s-t}{3} \\ s \\ t \end{pmatrix}$, so it has a basis of the vectors $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$ (corresponding to the choices $s = 1, t = 0$ and $s = 0, t = 1$ respectively), and after computing the reduced column echelon form and looking for missing pivots, we obtain a relative basis consisting of the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This vector gives rise to the thread f, $\varphi(f) = \begin{pmatrix} 8 \\ 23 \\ -1 \end{pmatrix}$, $\varphi^2(f) = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$. Since our space is 3-dimensional, this thread forms a basis.

Example 2. $V = \mathbb{R}^4$, φ is multiplication by the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$. In this case, $\varphi^2 = 0$, $\operatorname{rk}(\varphi) = 2$, $\operatorname{rk}(\varphi^k) = 0$ for $k \geqslant 2$, $\operatorname{null}(\varphi) = 2$, $\operatorname{null}(\varphi^k) = 4$ for $k \geqslant 2$. Moreover, $\operatorname{Ker}(\varphi) = \{\begin{pmatrix} -s \\ t \\ t \\ s \end{pmatrix}\}$. We have a sequence of subspaces $V = \operatorname{Ker}(\varphi^2) \supset \operatorname{Ker}(\varphi) \supset \{0\}$. The first one relative to the second one is two-dimensional ($\dim\operatorname{Ker}(\varphi^2) - \dim\operatorname{Ker}(\varphi) = 2$). Clearly, the vectors $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ (corresponding to $s = 1, t = 0$ and $s = 0, t = 1$ respectively) form a basis of the kernel of φ, and after computing the reduced column echelon form and looking for missing pivots, we obtain a relative basis consisting of the vectors $f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. These vectors give rise to threads f_1, $\varphi(f_1) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and f_2, $\varphi(f_2) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. These two threads together contain four vectors, and we have a basis.