1212: Linear Algebra II

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Lecture 1

Recollections from the module 1111

In this module, we shall be using extensively notions and methods from the module 1111. Please consult the notes for that module when you have questions. The most important notions are that of

- vector space
- linear map and linear transformation
- linearly independent set vectors
- spanning set of vectors
- basis, dimension
- coordinates relative to a basis
- transition matrix between two coordinate systems

You should know how to compute change of coordinate between different coordinate systems, and how to transform matrices of linear maps and linear transformations between different coordinate systems.

Outline of this module

The main question that we shall address this semester is

> Given a linear map or a linear transformation, how simple can its matrix be made by a change of coordinates?

We shall find an answer to this question, and see some applications of it.

The module roughly consists of three parts:

- simple observations for the general case of the question concerned
- answering the question in case there is some extra structure present (scalar product)
- answering the question in full generality ("Jordan decomposition theorem").

There will be seven tutorial classes during this semester, and a midterm exam on March 2.

Kernels and images

Recall that a *subspace* of a vector space V is a non-empty subset $W \subset V$ for which

- if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
- if $w \in W$ and c is a scalar, then $c \cdot w \in W$.

It is easy to see that a subspace of a vector space is a vector space with the same operations.

Definition 1. Let $\varphi\colon V \to W$ be a linear map between two vector spaces. We define its *kernel* $\mathrm{Ker}(\varphi)$ as the set of all vectors $v \in V$ for which $\varphi(v) = 0$, and its image $\mathrm{Im}(\varphi)$ as the set of all vectors $w \in W$ such that $w = \varphi(v)$ for some $v \in V$.

Lemma 1. *The subset* $\mathrm{Ker}(\varphi)$ *is a subset of* V*, and the subset* $\mathrm{Im}(\varphi)$ *is a subspace of* W.

Proof. It follows immediately from definitions and the linearity of φ:
if $v_1, v_2 \in \mathrm{Ker}(\varphi)$, then $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2) = 0 + 0 = 0$,
if $v \in \mathrm{Ker}(\varphi)$, then for any scalar c we have $\varphi(c \cdot v) = c\varphi(v) = c \cdot 0 = 0$,
if $w_1, w_2 \in \mathrm{Im}(\varphi)$, then $w_1 = \varphi(v_1)$ for some $v_1 \in V$ and $w_2 = \varphi(v_2)$ for some $v_2 \in V$, so we have $w_1 + w_2 = \varphi(v_1) + \varphi(v_2) = \varphi(v_1 + v_2)$,
if $w \in \mathrm{Im}(\varphi)$, then $w = \varphi(v)$ for some $v \in V$, so for any scalar c we have $c \cdot w = c\varphi(v) = \varphi(c \cdot v)$. $\square$

We shall discuss properties of kernels and images in our next class.