1212: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 5

In practice, it is important sometimes to determine the intersection of two subspaces, each presented as a linear span of several vectors. This question naturally splits into two different questions.

First, it makes sense to find a basis of each of these subspaces. To determine a basis for a linear span of given vectors, the easiest way is to form the matrix whose columns are the given vectors, and find its reduced column echelon form (like the reduced row echelon form, but with elementary operations on columns). Nonzero columns of the result form a basis of the linear span.

Once we know a basis $v_1, \ldots, v_k$ for the first subspace, and a basis $w_1, \ldots, w_l$ for the second one, the question reduces to solving the linear system $c_1v_1 + \ldots + c_kv_k = d_1w_1 + \ldots + d_lw_l$. For each solution to this system, the vector $c_1v_1 + \ldots + c_kv_k$ is in the intersection, and vice versa. Computationally, the first step does a part of the job for us, because computing the reduced column echelon form produces a system of equations with many zero entries already.

Example 1. Let us consider two following subspaces of $\mathbb{R}^5$: the subspace U is the span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 5 \\ -1 \\ -1 \\ 14 \end{pmatrix}$, and subspace W is the span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ -2 \\ -3 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$. We shall compute $\dim U$ and $\dim W$, and describe the intersection of $U \cap W$.

Let us first find the bases of these subspaces. As we mentioned, we should compute reduced column echelon forms, so in order to not introduce new notation, we shall temporarily transpose matrices, and write coordinates of all vectors in rows. To find a basis of the space spanned by rows of a matrix, we bring it to its reduced row echelon form; the nonzero rows of the result give us a basis. The reduced row echelon form of the matrix $\begin{pmatrix} 2 & 1 & 0 & -4 & 2 \\ -4 & 1 & 3 & -1 & 2 \\ 0 & 5 & -1 & -1 & 14 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 & -5/3 & -1/3 \\ 0 & 1 & 0 & -2/3 & 8/3 \\ 0 & 0 & 1 & -7/3 & -2/3 \end{pmatrix}$, and the reduced row echelon form of the matrix $\begin{pmatrix} 2 & 1 & 0 & 1 & 1 \\ 2 & -1 & -2 & -3 & -1 \\ 1 & 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1 & 7/3 \\ 0 & 0 & 1 & 1 & -4/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. This means that each of the subspaces is three-dimensional. To compute the intersection, recall that by definition the intersection consists of all vectors that belong to both of the subspaces. Let us denote by u_1, u_2, u_3 the basis vectors for U found above, and by w_1, w_2, w_3 the basis vectors for W found above. Then the intersection consists of all vectors v that can be represented in the form

$$v = c_1u_1 + c_2u_2 + c_3u_3 = c_4w_1 + c_5w_2 + c_6w_3$$

for some $c_1, c_2, c_3, c_4, c_5, c_6$, or, equivalently,

$$c_1u_1 + c_2u_2 + c_3u_3 - c_4w_1 - c_5w_2 - c_6w_3 = 0.$$

This is a homogeneous system of linear equations with unknowns c_i. Its matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -5/3 & -2/3 & -7/3 & 0 & -1 & -1 \\ -1/3 & 8/3 & -2/3 & 2/3 & -7/3 & 4/3 \end{pmatrix}$$

Bringing it to the reduced row echelon form (calculations are omitted), we get the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so the general solution is $c_1 = -c_5 - 2c_6$, $c_2 = c_5$, $c_3 = c_6$, $c_4 = -c_5 - 2c_6$, and the intersection can be described as the set of all vectors of the form $(-c_5 - 2c_6)w_1 + c_5 w_2 + c_6 w_3 = c_5(w_2 - w_1) + c_6(w_3 - 2w_1)$, so the vectors $w_2 - w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}$ and $w_3 - 2w_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ form a basis of the intersection.

Invariant subspaces

Definition 1. Let $\varphi\colon V \to V$ be a linear transformation. A subspace U of V is said to be *invariant* under φ if $\varphi(U) \subset (U)$.

Example 2. All multiples of an eigenvector of φ form a subspace of V that is invariant under φ. Indeed, all multiples of any vector form a subspace, and if it is an eigenvector, then φ maps any vector from this subspace to its multiple.

Let us prove an important result that utilises the notion of an invariant subspace. As we mentioned before, we assume that complex numbers are used as scalars.

Two linear transformations φ and ψ are said to *commute* if $\varphi \circ \psi = \psi \circ \varphi$, so that the result of consecutive application of φ and ψ does not depend on the order in which they are applied.

Theorem 1. *Any set of pairwise commuting operators $\varphi_i\colon V \to V$ has a common eigenvector.*

Proof. We shall prove it by induction on $\dim(V)$. If $\dim(V) = 1$, then any basis vector of V is a common eigenvector for these operators. Assume the statement is proved for $\dim(V) = k$. Let us prove it for $\dim(V) = k + 1$. If all the operators φ_i are scalar multiples of the identity map, that is there exist scalars c_i such that for all v and all i we have $\varphi_i(v) = c_i \cdot v$, then every non-zero vector is a common eigenvector of these transformations. Suppose that for some i the operator φ_i is not a scalar multiple of the identity map. Let us consider some eigenvalue λ of φ_i, and consider the solution space to the system of equations $\varphi_i(v) = \lambda \cdot v$. This solution space is a subspace W with $0 < \dim(W) < \dim(V)$. Let us note that W is an invariant subspace of all our transformations: if $w \in W$, and $w' = \varphi_k(w)$, we have

$$\varphi_i(w') = \varphi_i(\varphi_k(w)) = \varphi_k(\varphi_i(w)) = \varphi_k(\lambda \cdot w) = \lambda \varphi_k(w) = \lambda \cdot w',$$

so $w' \in W$. By induction, there is a common eigenvector in W, as required. $\square$