Solutions to midterm test

1. (**a**) A linear map from a vector space V to a vector space W is a function $\varphi\colon V \to W$ satisfying the conditions $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$ and $\varphi(c \cdot v) = c \cdot \varphi(v)$ for all vectors v, v_1, v_2 and all scalars c.

A linear transformation of a vector space V is a linear map from V to V.

A subspace U of a vector space V is said to be invariant under a linear transformation φ, if $\varphi(U) \subset U$, that is for all $u \in U$, $\varphi(u) \in U$.

(**b**) Solution: denote these vectors by v_1 and v_2; we have $Av_1 = \begin{pmatrix} -7 \\ 0 \\ 21 \\ -7 \end{pmatrix}$ and $Av_2 = \begin{pmatrix} -1 \\ -4 \\ -25 \\ 11 \end{pmatrix}$. Our subspace U is invariant if the image of every vector is again in U; it is enough to check that the images of v_1 and v_2 are in U, so we have to find out whether or not Av_i can be represented as combinations of v_1 and v_2. Solving the corresponding systems of equations, we get $Av_1 = 7v_1 + 7v_2$, $Av_2 = -7v_1 - 3v_2$, so U is invariant.

2. (**a**) A bilinear form on a real vector V is a function $f\colon V \times V \to \mathbb{R}$ satisfying the conditions $f(v_1+v_2, w) = f(v_1, w)+f(v_2, w)$, $f(v, w_1+w_2) = f(v, w_1)+f(v, w_2)$, $f(c\cdot v, w) = c\, f(v, w) = f(v, c\cdot w)$ for all vectors v, v_1, v_2, w, w_1, w_2, and all scalars c.

A symmetric bilinear form f is said to be positive definite if $f(v, v) > 0$ for $v \neq 0$.

(**b**) The associated bilinear form has the matrix

$$\begin{pmatrix} 3 & a & 1-a \\ a & a+2 & a \\ 1-a & a & 3 \end{pmatrix}.$$

By Sylvester's criterion, Q is positive definite if and only is the top left corner determinants $\Delta_1, \Delta_2, \Delta_3$ are positive. We have $\Delta_1 = 3$, $\Delta_2 = 3a+6-a^2$, $\Delta_3 = -3a^3-4a^2+12a+16 = -(a-2)(a+2)(3a+4)$. We have $\Delta_2 > 0$ for $\frac{3-\sqrt{33}}{2} < a < \frac{3+\sqrt{33}}{2}$, and $\Delta_3 > 0$ for $a < -2$ and for $-4/3 < a < 2$. We have $-2 < \frac{3-\sqrt{33}}{2} < -4/3$, and $2 < \frac{3+\sqrt{33}}{2}$, so the answer is $-4/3 < a < 2$.

3. (**a**) A Euclidean vector space is a vector space equipped with a symmetric positive definite bilinear form. (The value of that form on vectors v and w is denoted by (v, w)). A basis $e_1, \ldots, e_n$ of V is said to be orthogonal if $(e_i, e_j) = 0$ for $i \neq j$. An orthogonal basis is said to be orthonormal if in addition $(e_i, e_i) = 1$ for all $i = 1, \ldots, n$.

(**b**) Assembling these vectors in a matrix, we obtain the matrix $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 2 & 3 & 4 \end{pmatrix}$, whose determinant is equal to 15, so it is invertible, and hence the columns of this matrix form a basis.

(**c**) We compute

$$e_1 = f_1, e_2 = f_2 - \frac{(e_1, f_2)}{(e_1, e_1)} e_1, e_3 = f_3 - \frac{(e_1, f_3)}{(e_1, e_1)} e_1 - \frac{(e_2, f_3)}{(e_2, e_2)} e_2,$$

that is

$$e_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, e_2 = \begin{pmatrix} 6/5 \\ -2 \\ 3/5 \end{pmatrix}, e_3 = \begin{pmatrix} 60/29 \\ 45/29 \\ 30/29 \end{pmatrix}.$$