MA 1212: Linear Algebra II
Tutorial problems, February 12, 2015

1. $\Delta_1 = 3$, $\Delta_2 = 2$, $\Delta_3 = -7$, , so by Jacobi's theorem the signature can be read from the sequence $1/3$, $3/2$, $-2/7$; it is $(2, 1, 0)$.

2. The matrix of the corresponding bilinear form is

$$A = \begin{pmatrix} 18+a & 5 & -a-4 \\ 5 & 3 & -2 \\ -a-4 & -2 & a \end{pmatrix}.$$

We have $\Delta_1 = 18 + a$, $\Delta_2 = 3a + 29$, $\Delta_3 = 21a - 40$. All these numbers are positive if and only if

$$a > \frac{40}{21}.$$

3. The characteristic polynomial of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is $2t - t^3 = t(2 - t^2)$, so the eigenvalues of this matrix are 0 and $\pm\sqrt{2}$. The matrix of the bilinear form corresponding to the quadratic form

$$q(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_2 + x_2x_3$$

is equal to $1/2$ of the matrix in question, so its signature can be read off the eigenvalues of this matrix, and is $(1, 1, 1)$.

4. We have

$$\frac{\partial\varphi}{\partial x_1} = 2\sin(x_1 - x_2)\cos(x_1 - x_2) - (x_1 + 2cx_2)e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2} = \\ = \sin 2(x_1 - x_2) - (x_1 + 2cx_2)e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2},$$

$$\frac{\partial\varphi}{\partial x_2} = -2\sin(x_1 - x_2)\cos(x_1 - x_2) - (2cx_1 + x_2)e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2} = \\ = -\sin 2(x_1 - x_2) - (2cx_1 + x_2)e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2},$$

and

$$\frac{\partial^2\varphi}{\partial x_1^2} = 2\cos 2(x_1 - x_2) - e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2} - (x_1 + 2cx_2)^2 e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2},$$

$$\frac{\partial^2\varphi}{\partial x_1 \partial x_2} = -2\cos 2(x_1 - x_2) - 2ce^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2} - (x_1 + 2cx_2)(2cx_1 + x_2)e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2},$$

$$\frac{\partial^2\varphi}{\partial x_2^2} = 2\cos 2(x_1 - x_2) - e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2} - (2cx_1 + x_2)^2 e^{\frac{1}{2}(x_1^2+x_2^2)+2cx_1x_2},$$

so the matrix A is

$$\begin{pmatrix} 1 & -2-2c \\ -2-2c & 1 \end{pmatrix}.$$

By Sylvester's criterion, this quadratic form is positive definite if and only if $\Delta_2 = 1 - (2+2c)^2 > 0$ (since $\Delta_1 = 1$). We have

$$1 - (2+2c)^2 = (1+2+2c)(1-2-2c) = (3+2c)(-1-2c),$$

so the quadratic form is positive definite for $-3/2 < c < -1/2$. In particular, this holds for $c = -3/5$.