MA 1212: Linear Algebra II
Tutorial problems, March 19, 2015

1. The characteristic polynomial of A is $t^2-2t+1=(t-1)^2$, so the only eigenvalue is 1. We have $A-I=\begin{pmatrix}2&-1\\4&-2\end{pmatrix}$, and this matrix evidently is of rank 1. Also, $(A-I)^2=0$, so there is a stabilising sequence of subspaces $\mathrm{Ker}(A-I)\subset\mathrm{Ker}(A-I)^2=V$. The dimension gap between these is equal to 1, and we have to find a relative basis. The kernel of $A-I$ is spanned by the vector $\begin{pmatrix}1\\2\end{pmatrix}$, and for the relative basis we can take the vector $f=\begin{pmatrix}0\\1\end{pmatrix}$ which makes up for missing pivot. This vector gives rise to a thread $f,(A-I)f=\begin{pmatrix}-1\\-2\end{pmatrix}$, and reversing the order of vectors in the thread we get a Jordan basis $f,(A-I)f$, and the Jordan normal form $\begin{pmatrix}1&0\\1&1\end{pmatrix}$.

2. The characteristic polynomial of A is $\det(A-tI)=-t^3+6t^2-12t+8=(2-t)^3$, so the only eigenvalue is equal to 2. Furthermore, $A-2I=\begin{pmatrix}4&5&-2\\-8&-10&4\\-12&-15&6\end{pmatrix}$, $(A-2I)^2=0$, $\mathrm{rk}(A-2I)=1$, $\mathrm{rk}((A-2I)^2)=0$. Thus, we have a sequence of subspaces $\mathrm{Ker}(A-2I)\subset\mathrm{Ker}(A-2I)^2=\mathrm{Ker}(A-2I)^3=\ldots=V$. The kernel of $A-2I$ is two-dimensional, and consists of all vectors $\begin{pmatrix}x\\y\\z\end{pmatrix}$ satisfying $4x+5y-2z=0$, therefore y and z are free variables, and we have a basis of the kernel that consists of the vectors $\begin{pmatrix}-5/4\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1/2\\0\\1\end{pmatrix}$. Bringing the matrix whose columns are these vectors to its reduced column echelon form, we observe that the missing pivot is the one in the third row, so the vector $e=\begin{pmatrix}0\\0\\1\end{pmatrix}$ forms a basis of V relative to the kernel. We have $(A-2I)e=\begin{pmatrix}-2\\4\\6\end{pmatrix}$. This vector belongs to the kernel, and we should find a basis of the kernel relative to the span of this vector. We reduce the basis vectors of the kernel using this vector:

$$\begin{pmatrix}-2&-5/4&1/2\\4&1&0\\6&0&1\end{pmatrix}\to\begin{pmatrix}-2&0&0\\4&-3/2&1\\6&-15/4&5/2\end{pmatrix},$$

and see that the both the second and the third column are proportional to the vector $f=\begin{pmatrix}0\\2\\5\end{pmatrix}$, which gives rise to a thread of length 1 and completes the basis. Overall, a Jordan basis is given by $e,(A-2I)e,f$, and the Jordan normal form has a block of size 2 with 2 on the diagonal and a block of size 1 with 2 on the diagonal.

3. The characteristic polynomial of A is $\det(A-tI)=-t^3-t^2+t+1=(1-t)(1+t)^2$, so the eigenvalues are 1 and -1. Furthermore, $\mathrm{rk}(A-I)=2$, $\mathrm{rk}(A-I)^2=2$, $\mathrm{rk}(A+I)=2$, $\mathrm{rk}(A+I)^2=1$.

Thus, the kernels of powers of $A - I$ stabilise instantly, so we should expect a thread of length 1 for the eigenvalue 1, whereas the kernels of powers of $A + I$ do not stabilise for at least two steps, so that would give a thread of length at least 2, hence a thread of length 2 because our space is 3-dimensional. To determine the basis of $\text{Ker}(A - I)$, we solve the system $(A - I)v = 0$ and obtain a vector $f = \begin{pmatrix} -6 \\ 4 \\ 1 \end{pmatrix}$. To deal with the eigenvalue -1, we see that the kernel of $A + I$ is spanned by the vector $\begin{pmatrix} -4 \\ 5/2 \\ 1 \end{pmatrix}$, the kernel of $(A + I)^2 = \begin{pmatrix} -24 & -48 & 24 \\ 16 & 32 & -16 \\ 4 & 8 & -4 \end{pmatrix}$ is spanned by the vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Reducing the latter vectors using the former one, we end up with the vector $e = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, which gives rise to a thread $e, (A + I)e = \begin{pmatrix} 64 \\ 40 \\ -16 \end{pmatrix}$. Overall, a Jordan basis is given by $f, e, (A + I)e$, and the Jordan normal form has a block of size 2 with -1 on the diagonal, and a block of size 1 with 1 on the diagonal.