MA 1111: Linear Algebra I
Selected answers/solutions to the assignment due December 17, 2015

1. (**a**) We have

$$\begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} b_n \\ 3b_n - b_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} b_{n-1} \\ b_n \end{pmatrix},$$

and iterating that, we get

$$\begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(**b**) Eigenvalues are roots of $t^2 - 3t + 1 = 0$, i.e. $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$. Clearly, $C = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$ is the transition matrix from the basis of standard unit vectors to the basis of eigenvectors. Then in the basis of eigenvectors we obtain the matrix $C^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix} C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, so

$$\begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = C \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n C^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \\ \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} \end{pmatrix},$$

and

$$b_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}} \left(\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right).$$

2. We have $\det(A - cI_3) = -c^3 + 4c^2 - 5c + 2 = -(c-1)^2(c-2)$, so the eigenvalues of this matrix are 1 and 2. Solving the systems of equations $Ax = x$ and $Ax = 2x$, we see that each solution is proportional to, respectively, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, so there is no basis of eigenvectors, and the answer is "no".

3. This can be done by a direct computation: let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A^2 - \mathrm{tr}(A) \cdot A + \det(A) \cdot I_2 =$$
$$= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{pmatrix} - \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = 0.$$

4. Note that $\det(A^3) = (\det(A))^3$, so if $A^3 = 0$, then $\det(A) = 0$. According to the previous question, we then have $A^2 - \mathrm{tr}(A)A = 0$, so $A^2 = \mathrm{tr}(A)A$. If $\mathrm{tr}(A) = 0$, we conclude that $A^2 = 0$. Otherwise, we have

$$0 = A^3 = A^2 \cdot A = \mathrm{tr}(A)A \cdot A = \mathrm{tr}(A)A^2 = (\mathrm{tr}(A))^2 \cdot A,$$

which for $\mathrm{tr}(A) \neq 0$ implies $A = 0$.