

MA 1111: Linear Algebra I  
Selected answers/solutions to the assignment due October 15, 2015

1. Let us solve this system of equations by usual methods. The corresponding matrix is

$$\begin{pmatrix} 1 & -1 & 3 & -1 \\ 4 & -1 & 1 & 2 \end{pmatrix},$$

and it is brought to its reduced row echelon form as follows:

$$\begin{pmatrix} 1 & -1 & 3 & -1 \\ 4 & -1 & 1 & 2 \end{pmatrix} \xrightarrow{(2)-4(1)} \begin{pmatrix} 1 & -1 & 3 & -1 \\ 0 & 3 & -11 & 6 \end{pmatrix} \xrightarrow{(1)+1/3(2), 1/3(2)} \begin{pmatrix} 1 & 0 & -2/3 & 1 \\ 0 & 1 & -11/3 & 2 \end{pmatrix}.$$

This shows that  $z$  is the free unknown of this system. Assigning an arbitrary value  $t$  to it, we get a parametric representation of the solution set

$$\begin{cases} x = 1 + 2/3t, \\ y = 2 + 11/3t, \\ z = t. \end{cases}$$

2. The matrix corresponding to this system of equations is

$$\begin{pmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ -3 & -4 & -2 & 0 & 1 & -1 \\ 2 & 3 & 2 & 3 & 2 & -2 \end{pmatrix}.$$

Let us compute its reduced row echelon form. Using the pivot of the first row, we cancel other elements of the first column:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ -3 & -4 & -2 & 0 & 1 & -1 \\ 2 & 3 & 2 & 3 & 2 & -2 \end{pmatrix} \xrightarrow{(2)+3(1), (3)-2(1)} \begin{pmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 6 & -2 & -1 \\ 0 & 1 & 0 & -1 & 4 & -2 \end{pmatrix}.$$

Next, we use the first nonzero entry of the second row to cancel other elements of the first column, and multiply the second row by  $-1$  to make its first nonzero entry a pivot:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 6 & -2 & -1 \\ 0 & 1 & 0 & -1 & 4 & -2 \end{pmatrix} \xrightarrow{(3)+(2), (1)+(2), (-1)\times(2)} \begin{pmatrix} 1 & 0 & 2 & 8 & -3 & -1 \\ 0 & 1 & -1 & -6 & 2 & 1 \\ 0 & 0 & 1 & 5 & 2 & -3 \end{pmatrix}.$$

Finally, we use the pivot of the third row to cancel other elements in the first column:

$$\begin{pmatrix} 1 & 0 & 2 & 8 & -3 & -1 \\ 0 & 1 & -1 & -6 & 2 & 1 \\ 0 & 0 & 1 & 5 & 2 & -3 \end{pmatrix} \xrightarrow{(1)-2(3), (2)+(3)} \begin{pmatrix} 1 & 0 & 0 & -2 & -7 & 5 \\ 0 & 1 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & 5 & 2 & -3 \end{pmatrix}.$$

We observe that  $x_1$ ,  $x_2$ , and  $x_3$  are principal (pivotal) unknowns, and  $x_4$ ,  $x_5$  are free unknowns. Assigning arbitrary values  $s$  and  $t$  to those unknowns, we obtain a parametric

representation of the solution set:

$$\begin{cases} x_1 = 2s + 7t + 5, \\ x_2 = s - 4t - 2, \\ x_3 = -5s - 2t - 3, \\ x_4 = s, \\ x_5 = t. \end{cases}$$

3. (a)

$$\begin{cases} x_1 + 4x_2 + 5x_3 + x_4 = 1, \\ -2x_1 - 2x_2 - 3x_3 - 3x_4 = -4, \\ x_1 - 2x_2 - 2x_3 + 2x_4 = -4. \end{cases}$$

(b) Let us use the pivot of the first row of the matrix to cancel other entries in the first column:

$$\begin{pmatrix} 1 & 4 & 5 & 1 & 1 \\ -2 & -2 & -3 & -3 & -4 \\ 1 & -2 & -2 & 2 & -4 \end{pmatrix} \xrightarrow{(2)+2(1), (3)-(1)} \begin{pmatrix} 1 & 4 & 5 & 1 & 1 \\ 0 & 6 & 7 & -1 & -2 \\ 0 & -6 & -7 & 1 & -3 \end{pmatrix}.$$

Now, we use the first nonzero entry of the second row to cancel other entries in the second column, and multiply the second row by  $1/6$  to make its first nonzero entry a pivot:

$$\begin{pmatrix} 1 & 4 & 5 & 1 & 1 \\ 0 & 6 & 7 & -1 & -2 \\ 0 & -6 & -7 & 1 & -3 \end{pmatrix} \xrightarrow{(1)-2/3(2), (3)+(2), 1/6 \times (2)} \begin{pmatrix} 1 & 0 & 1/3 & 5/3 & 7/3 \\ 0 & 1 & 7/6 & -1/6 & -1/3 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}.$$

Finally, we use the first nonzero entry of the third row to cancel other entries of its column, and re-scale the third row to create a pivot:

$$\begin{pmatrix} 1 & 0 & 1/3 & 5/3 & 7/3 \\ 0 & 1 & 7/6 & -1/6 & -1/3 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix} \xrightarrow{(1)+7/15(3), (2)-1/15(3), -1/5 \times (3)} \begin{pmatrix} 1 & 0 & 1/3 & 5/3 & 0 \\ 0 & 1 & 7/6 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We observe that the last row of the reduced row echelon form corresponds to the equation  $0 = 1$ , so the corresponding system of equations has no solutions.

4. (a) Let us first use the pivot of the first row to cancel the other entries of the corresponding column:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 & 2 & 5 \\ -1 & 2 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{(2)-(1), (3)-2(1), (4)+(1)} \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 & -2 & -1 & 1 \\ 0 & -1 & -3 & -1 & -3 & 0 & 3 \\ 0 & 3 & 3 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Next, we swap the second and the third row to take care of the fact that the first nonzero entry of the second row is to the right from the first nonzero entry of the third row:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 0 & -2 & -1 & 1 \\ 0 & -1 & -3 & -1 & -3 & 0 & 3 \\ 0 & 3 & 3 & 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{(2) \leftrightarrow (3)} \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & -1 & -3 & -1 & -3 & 0 & 3 \\ 0 & 0 & -1 & 0 & -2 & -1 & 1 \\ 0 & 3 & 3 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

Now, we use the first nonzero entry of the second row to cancel other entries of its column, and re-scale the second row to create a pivot:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & -1 & -3 & -1 & -3 & 0 & 3 \\ 0 & 0 & -1 & 0 & -2 & -1 & 1 \\ 0 & 3 & 3 & 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{(1)+(2), (4)+3(2), -1 \times (2)} \begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 3 & 1 & 3 & 0 & -3 \\ 0 & 0 & -1 & 0 & -2 & -1 & 1 \\ 0 & 0 & -6 & -1 & -7 & 2 & 11 \end{pmatrix}.$$

Next, we use the first nonzero entry of the third row to cancel all other entries in its column, and re-scale the third row to create a pivot:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 3 & 1 & 3 & 0 & -3 \\ 0 & 0 & -1 & 0 & -2 & -1 & 1 \\ 0 & 0 & -6 & -1 & -7 & 2 & 11 \end{pmatrix} \xrightarrow{(1)-(3), (2)+3(3), (4)-6(3), -1 \times (3)} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -1 & 5 & 8 & 5 \end{pmatrix}.$$

Finally, we use the first nonzero entry of the fourth row to cancel all other entries of its column, and re-scale the fourth row to create a pivot:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -1 & 5 & 8 & 5 \end{pmatrix} \xrightarrow{(2)+(4), -1 \times (4)} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 2 & 5 & 5 \\ 0 & 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -5 & -8 & -5 \end{pmatrix}.$$

(b) We observe that the first four unknowns are principal, and that the last two unknowns are free. Assigning arbitrary values  $s$  and  $t$  to the last two unknowns, we get a parametric representation of the solution set:

$$\begin{cases} x_1 = 3 - s - 2t, \\ x_2 = 5 - 2s - 5t, \\ x_3 = -1 - 2s - t, \\ x_4 = -5 + 5s + 8t, \\ x_5 = s, \\ x_6 = t. \end{cases}$$

5. Suppose that  $(\mathbf{u}_1, \mathbf{u}_2)$  and  $(\mathbf{v}_1, \mathbf{v}_2)$  are two different solutions, so that

$$\begin{cases} \mathbf{a}\mathbf{u}_1 + \mathbf{b}\mathbf{u}_2 = \mathbf{e}, \\ \mathbf{c}\mathbf{u}_1 + \mathbf{d}\mathbf{u}_2 = \mathbf{f} \end{cases}$$

and

$$\begin{cases} \mathbf{a}\mathbf{v}_1 + \mathbf{b}\mathbf{v}_2 = \mathbf{e}, \\ \mathbf{c}\mathbf{v}_1 + \mathbf{d}\mathbf{v}_2 = \mathbf{f}. \end{cases}$$

Then subtracting these, we get

$$\begin{cases} \mathbf{a}(\mathbf{u}_1 - \mathbf{v}_1) + \mathbf{b}(\mathbf{u}_2 - \mathbf{v}_2) = \mathbf{0}, \\ \mathbf{c}(\mathbf{u}_1 - \mathbf{v}_1) + \mathbf{d}(\mathbf{u}_2 - \mathbf{v}_2) = \mathbf{0} \end{cases}$$

which shows that  $(\mathbf{u}_1 - \mathbf{v}_1, \mathbf{u}_2 - \mathbf{v}_2)$  is a solution of the second system which is different from  $(\mathbf{0}, \mathbf{0})$ .