MA 1111: Linear Algebra I
Selected answers/solutions to the assignment due October 29, 2015

1. Let us note that in order for this matrix to be a matrix representing some permutation, we must have $\{k, l\} = \{4, 6\}$ and $\{i, j\} = \{2, 5\}$. Let us examine the choice $i = 2$, $j = 5$, $k = 4$, $l = 6$. The corresponding permutation is $\begin{pmatrix} 5 & 2 & 4 & 3 & 6 & 1 \\ 4 & 1 & 3 & 2 & 6 & 5 \end{pmatrix}$; it is even because the total number of inversions in the two rows is 14 (52, 54, 53, 51, 21, 43, 41, 31, 61 in the top row, and 41, 43, 42, 32, 65 in the bottom row). Therefore, for the two choices that correspond to swapping one of the pairs $i \leftrightarrow j$ and $k \leftrightarrow l$, the permutation is odd, and for the choice corresponding to swapping both, the permutation is even. *Answer:* $i = 2$, $j = 5$, $k = 4$, $l = 6$, and $i = 5$, $j = 2$, $k = 6$, $l = 4$.

2. (**a**) We have $\mathrm{tr}(A^{-1}BA) = \mathrm{tr}((A^{-1}B)A) = \mathrm{tr}(A(A^{-1}B)) = \mathrm{tr}(AA^{-1}B) = \mathrm{tr}(B)$.

(**b**) Assume that A is invertible. Multiplying by A^{-1} on the left, we get $B - A^{-1}BA = I_n$. Now, computing traces of both sides gives us $0 = n$, a contradiction.

3. (**a**) Performing elementary row operations, we get

$$\det\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 3 & 9 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & -6 \end{pmatrix} = -6$$

(**b**) Performing elementary row operations, we get

$$\det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 2 & 0 & 3 & -1 \\ 4 & 2 & 3 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & -2 & 11 & 5 \\ 0 & -3 & 6 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -9/2 & 5/2 \end{pmatrix} =$$
$$= 4/2\det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -9 & 5 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 1 & -2 & -1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 14 \end{pmatrix} = -56.$$

4. (**a**) We have $\det(A) = (2-c)^2 - 1 = c^2 - 4c + 3 = (c-1)(c-3)$, so A is not invertible for $c = 1$ and for $c = 3$.

(**b**) Because of multilinearity, we have

$$\det(A) = \det\begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix} + \det\begin{pmatrix} 2 & 1 & 3 \\ c & c & 4c \\ 1 & 3 & -1 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix} + c\det\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ 1 & 3 & -1 \end{pmatrix}.$$

We compute

$$\det\begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix} \overset{(1)+3(3)}{=} \det\begin{pmatrix} 5 & 10 & 0 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix} = 5\det\begin{pmatrix} 1 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix} \overset{(1)-(2)}{=} 5\det\begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix}$$

concluding with $\det\begin{pmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix} = -20$, and

$$\det\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 4 \\ 1 & 3 & -1 \end{pmatrix} \overset{(1)-2(2),(3)-(2)}{=} \det\begin{pmatrix} 0 & -1 & -5 \\ 1 & 1 & 4 \\ 0 & 2 & -5 \end{pmatrix} \overset{(3)+2(1),(1)\leftrightarrow(2)}{=} -\det\begin{pmatrix} 1 & 1 & 4 \\ 0 & -1 & -5 \\ 0 & - & -15 \end{pmatrix}$$

concluding with $\det\begin{pmatrix} 2 & 1 & 3 \\ c & c & 4c \\ 1 & 3 & -1 \end{pmatrix} = -15c$. Thus, $\det(A) = -15c - 20$. We have $\det(A) = 0$ for $c = -20/15 = -4/3$.

5. Statements 1,2,5 are true, Statements 3,4 are false.

Indeed, we know from class that a matrix A is invertible if and only if $\det(A) \neq 0$. Since we have $\det(AB) = \det(A)\det(B)$, we conclude that AB is invertible if and only if both A and B is invertible, that is Statements 1 and 2. (Statement 2 was also proved directly in class).

For Statement 5, assume that both A and B are invertible. Then, by Statement 2, AB is invertible, which is a contradiction.

As a counterexample to Statement 3, take the matrix $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ that we discussed in class. We have $U^2 = U \cdot U = 0$, and $U \neq 0$.

As a counterexample to Statement 4, take $A = 0$ and $B = I_n$. Then $AB = 0$, but B is invertible.