1111: Linear Algebra I

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Lecture 20

Linear maps and matrix products

Last time we stated the following lemma which we shall now prove.

Lemma 1. *Let* U*,* V*, and* W *be vector spaces, and let* $\varphi\colon U \to V$ *and* $\psi\colon V \to W$ *be linear map. Then* $\psi \circ \varphi\colon u \mapsto \psi(\varphi(u))$ *is a linear map. Also, if* $e_1, \dots, e_n$*,* $f_1, \dots, f_m$*, and* $g_1, \dots, g_k$ *are bases of* U*,* V*, and* W *respectively, then*

$$A_{\psi\circ\varphi,\mathbf{e},\mathbf{g}} = A_{\psi,\mathbf{f},\mathbf{g}} A_{\varphi,\mathbf{e},\mathbf{f}}.$$

Proof. First, let us note that

$$(\psi \circ \varphi)(u_1 + u_2) = \psi(\varphi(u_1 + u_2)) = \psi(\varphi(u_1) + \varphi(u_2)) = \psi(\varphi(u_1)) + \psi(\varphi(u_2)) = (\psi \circ \varphi)(u_1) + (\psi \circ \varphi)(u_2),$$
$$(\psi \circ \varphi)(c \cdot u) = \psi(\varphi(c \cdot u)) = \psi(c\varphi(u)) = c\psi(\varphi(u)) = c(\psi \circ \varphi)(u),$$

so $\psi \circ \varphi$ is a linear map.

Let us prove the second statement. We take a vector $\mathbf{u} \in U$, and apply the formula of Lemma 2. On the one hand, we have

$$(\psi \circ \varphi(\mathbf{u}))_{\mathbf{g}} = A_{\psi\circ\varphi,\mathbf{e},\mathbf{g}} \mathbf{u}_{\mathbf{e}}.$$

On the other hand, we obtain,

$$(\psi \circ \varphi(\mathbf{u}))_{\mathbf{g}} = (\psi(\varphi(\mathbf{u})))_{\mathbf{g}} = A_{\psi,\mathbf{f},\mathbf{g}}(\varphi(\mathbf{u})_{\mathbf{f}}) = A_{\psi,\mathbf{f},\mathbf{g}}(A_{\varphi,\mathbf{e},\mathbf{f}} \mathbf{u}_{\mathbf{e}}) = (A_{\psi,\mathbf{f},\mathbf{g}} A_{\varphi,\mathbf{e},\mathbf{f}}) \mathbf{u}_{\mathbf{e}}.$$

Therefore

$$A_{\psi\circ\varphi,\mathbf{e},\mathbf{g}} \mathbf{u}_{\mathbf{e}} = (A_{\psi,\mathbf{f},\mathbf{g}} A_{\varphi,\mathbf{e},\mathbf{f}}) \mathbf{u}_{\mathbf{e}}$$

for every $u_{\mathbf{e}}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so

$$A_{\psi\circ\varphi,\mathbf{e},\mathbf{g}} = A_{\psi,\mathbf{f},\mathbf{g}} A_{\varphi,\mathbf{e},\mathbf{f}},$$

as required. □

Linear maps and change of coordinates

Now let us exhibit how matrices of linear maps transform under changes of coordinates.

Lemma 2. *Let* $\varphi\colon V \to W$ *be a linear map, and suppose that* $e_1, \dots, e_n$ *and* $e'_1, \dots, e'_n$ *are two bases of* V*, and* $f_1, \dots, f_m$ *and* $f'_1, \dots, f'_m$ *are two bases of* W*. Then*

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'} = M^{-1}_{\mathbf{f},\mathbf{f}'} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}.$$

Proof. Let us take a vector $\mathbf{v} \in V$. On the one hand, from the fundamental property of matrices of linear maps we know that

$$(\varphi(\mathbf{v}))_{\mathbf{f}'} = A_{\varphi,\mathbf{e}',\mathbf{f}'}\mathbf{v}_{\mathbf{e}'}.$$

On the other hand, applying results on transition matrices alongside with that fundamental property, we have

$$(\varphi(\mathbf{v}))_{\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}}(\varphi(\mathbf{v})_{\mathbf{f}}) = M_{\mathbf{f}',\mathbf{f}}(A_{\varphi,\mathbf{e},\mathbf{f}}\mathbf{v}_{\mathbf{e}}) = M_{\mathbf{f}',\mathbf{f}}(A_{\varphi,\mathbf{e},\mathbf{f}}(M_{\mathbf{e},\mathbf{e}'}\mathbf{v}_{\mathbf{e}'})) = (M_{\mathbf{f}',\mathbf{f}}A_{\varphi,\mathbf{e},\mathbf{f}}M_{\mathbf{e},\mathbf{e}'})\mathbf{v}_{\mathbf{e}'}.$$

Therefore,

$$A_{\varphi,\mathbf{e}',\mathbf{f}'}\mathbf{v}_{\mathbf{e}'} = (M_{\mathbf{f}',\mathbf{f}}A_{\varphi,\mathbf{e},\mathbf{f}}M_{\mathbf{e},\mathbf{e}'})\mathbf{v}_{\mathbf{e}'}$$

for every $\mathbf{v}_{\mathbf{e}'}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = (M_{\mathbf{f}',\mathbf{f}}A_{\varphi,\mathbf{e},\mathbf{f}}M_{\mathbf{e},\mathbf{e}'}) = (M_{\mathbf{f},\mathbf{f}'}^{-1}A_{\varphi,\mathbf{e},\mathbf{f}}M_{\mathbf{e},\mathbf{e}'})$$

because of properties of transition matrices proved earlier. □

Remark 1. Our formula

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}}A_{\varphi,\mathbf{e},\mathbf{f}}M_{\mathbf{e},\mathbf{e}'}.$$

shows that changing from the coordinate systems $\mathbf{e}, \mathbf{f}$ to *some* other coordinate system amounts to multiplying the matrix $A_{\varphi,\mathbf{e},\mathbf{f}}$ by some invertible matrices on the left and on the right, so effectively to performing a certain number of elementary row and column operations on this matrix. By such transformations, every matrix can be first brought to its reduced row echelon form by row operations, and then columns with pivots can be moved to the beginning and used to cancel all elements in non-pivotal columns.

The previous observation is very useful, but not directly applicable to linear transformations, as we shall now see.

Remark 2. For a linear transformation, it makes sense to use the same coordinate system for the input and the output. By definition, the matrix of a linear operator $\varphi\colon V \to V$ relative to the basis $e_1, \ldots, e_n$ is

$$A_{\varphi,\mathbf{e}} := A_{\varphi,\mathbf{e},\mathbf{e}}.$$

Lemma 3. *For a linear transformation $\varphi\colon V \to V$, and two bases $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ of V, we have*

$$A_{\varphi,\mathbf{e}'} = M_{\mathbf{e},\mathbf{e}'}^{-1}A_{\varphi,\mathbf{e}}M_{\mathbf{e},\mathbf{e}'}.$$

Proof. This is a particular case of Lemma 1. □