1111: Linear Algebra I

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Lecture 21

Linear transformations and change of coordinates

Recall that for a linear transformation $\varphi\colon V \to V$, and two bases $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ of V, we have

$$A_{\varphi,\mathbf{e}'} = M^{-1}_{\mathbf{e},\mathbf{e}'} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{e}'}.$$

This result shows that for a square matrix A, the change $A \mapsto C^{-1}AC$ with an invertible matrix C, corresponds to the situation where A is viewed as a matrix of a linear transformation, and C is viewed as a transition matrix for a coordinate change. You verified in your earlier home assignments that $\mathrm{tr}(C^{-1}AC) = \mathrm{tr}(A)$ and $\det(C^{-1}AC) = \det(A)$; these properties imply that the trace and the determinant do not depend on the choice of coordinates, and hence reflect some geometric properties of a linear transformation.

In case of the determinant, those properties have been hinted at in our previous classes: determinants compute how a linear transformation changes volumes of solids. In the case of the trace, the situation is a bit more subtle: the best one can get is a formula

$$\det(I_n + \varepsilon A) \approx 1 + \varepsilon\,\mathrm{tr}(A),$$

where $\approx$ means that the correction term is a polynomial expression in ε of magnitude bounded by a constant multiple of ε^2 (for small ε).

Example of change of coordinates

Example 1. Let us take two bases of $\mathbb{R}^2$: $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $f_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $f_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Suppose that the matrix of a linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ relative to the first basis is $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. Let us compute its matrix relative to the second basis. For that, we first compute the transition matrix $M_{\mathbf{e},\mathbf{f}}$. We have

$$f_1 = \begin{pmatrix} 7 \\ 5 \end{pmatrix} = 5e_1 + 2e_2,$$

$$f_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 3e_1 + e_2,$$

so

$$M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix},$$

and

$$M^{-1}_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}.$$

Therefore

$$A_{\varphi,\mathbf{f}} = M^{-1}_{\mathbf{e},\mathbf{f}} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 10 & 9 \end{pmatrix}.$$

Observe that the trace and the determinant indeed have not changed.

In this example, one of the two matrices is much simpler than the other ones: the first matrix is diagonal, which simplifies all sorts of computation. We already discussed last time that for linear transformations we cannot simplify a matrix as much as for linear maps. It is natural to ask whether we may make the corresponding matrix diagonal. Let us show that it is not possible in general. Consider the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If there exists an invertible matrix C for which $C^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}C = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, we can compare traces on the right and on the left, getting $a_1 + a_2 = 2$, $a_1 a_2 = 1$, so a_1 and a_2 are roots of the equation $x^2 - 2x + 1 = 0$, that is $a_1 = a_2 = 1$. But the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix, and it represents the identity transformation, so it is the same in any coordinate system, a contradiction.

However, this is the only source of that kind of obstacles. Let us consider one convincing example.

Computing Fibonacci numbers

Fibonacci numbers are defined recursively: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 2$, so that this sequence starts like this:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

We shall now explain how to derive a formula for these using linear algebra.

Idea 1: let us consider a much simpler question: let $g_0 = 1$, and $g_n = a g_{n-1}$ for $n \geqslant 1$. Then of course $g_n = a^n$.

In our case, each of the numbers is determined by two previous ones, let us store pairs! We put $v_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$.

Then

$$v_{n+1} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n \\ f_n + f_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} v_n,$$

therefore

$$v_{n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} v_n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} v_{n-1} = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1} v_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, we shall be able to compute Fibonacci numbers if we can compute the n-th power of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Suppose that our matrix, after a change of coordinates, becomes a diagonal matrix with a_1 and a_2 on the diagonal. Arguing as before, $a_1 + a_2 = 1$, $a_1 a_2 = -1$, so a_1 and a_2 are roots of $x^2 - x - 1 = 0$, that is $\frac{1 \pm \sqrt{5}}{2}$. How to find an appropriate basis, if possible?

Idea 2: What does it mean for a matrix of a linear operator φ to be diagonal in the system of coordinates given by the basis e_1, e_2? This means $\varphi(e_1) = a_1 e_1$, $\varphi(e_2) = a_2 e_2$. We shall utilize this in the next lecture.