1111: Linear Algebra I

Dr. Vladimir Dotsenko (Vlad)

Lecture 23

Eigenvalues and eigenvectors

Last time, when talking about Fibonacci numbers, we touched important notions of linear algebra: eigenvalues and eigenvectors. If V is a vector space, $\varphi \colon V \to V$ is a linear transformation, and $\mathbf{v} \neq 0$ a vector for which $\varphi(\mathbf{v})$ is proportional to $\mathbf{v}$, that is $\varphi(\mathbf{v}) = c \cdot \mathbf{v}$ for some c, then $\mathbf{v}$ is called an *eigenvector* of φ, and c is the corresponding *eigenvalue*.

Usually, one first finds all eigenvalues, and then the corresponding eigenvectors. If φ is represented by an $n \times n$-matrix $A = A_{\varphi, \mathbf{e}}$, then eigenvalues are roots of the equation $\det(A - cI_n) = 0$. The left hand side of this equation is a degree n polynomial in c. Then eigenvectors correspond to solutions to the system of equations $(A - c_0 I_n)x = 0$, where c_0 is an eigenvalue.

Theorem 1. *If A has n distinct eigenvalues, then the corresponding eigenvectors form a basis of V.*

We shall prove this result in the next semester, it is not too difficult.

If eigenvalues are not distinct, everything can become more complicated.

Example 1. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We have $\det(A - cI_2) = (1 - c)^2$, so the only eigenvalue is 1. We have $Ax = x$ for every vector x, so any basis is a basis of eigenvectors.

Example 2. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We have $\det(A - cI_2) = c^2$, so the only eigenvalue is 0. Moreover, all eigenvectors are scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so in this case there is no basis of eigenvectors.

Example 3. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have $\det(A - cI_2) = c^2 + 1$, so if we use real scalars, then there are no eigenvalues, and if scalars are complex, the eigenvalues are $\pm i$, and there is a basis consisting of eigenvectors.

A toy example: "porridge problem"

To give another example of how eigenvectors and eigenvalues can be useful, let us consider the following toy problem:

> Five not quite hungry but quite playful kids are sat down around a round table. There is a bowl of porridge in front of each of them, and they generally have different amounts of porridge. They decide to play a game. Every minute each of them simultaneously gives one half of what they have to each of their neighbours. What is the dynamics of this process? Will they have roughly the same amounts in a while, or will the amounts of porridge in bowls keep changing significantly?

Intuitively, one would guess that after a while porridge will be distributed almost equally because of ongoing "averaging". However, one has to be careful. For example, if there are four kids, and amounts of porridge around the circle (in some units) are $1, 0, 1, 0$, then after one averaging this becomes $0, 1, 0, 1$, and

after another averaging will be back to the original 1, 0, 1, 0, so the picture will oscillate between these two distributions!

A promising example: if in the beginning the amounts of porridge around the circle were 1, 1.5, 0.5, 0.3, 0.7, then after 100 iterations all the amounts become approximately 0.886666666 (simple numeric simulation), so the distribution gets equal.

How to deal with this question in general? By means of linear algebra! Let us record amounts of porridge in a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

Then after one step, the new distribution of porridge is given by the vector

$$\begin{pmatrix} \frac{x_2+x_5}{2} \\ \frac{x_1+x_3}{2} \\ \frac{x_2+x_4}{2} \\ \frac{x_3+x_5}{2} \\ \frac{x_4+x_1}{2} \end{pmatrix} = A \cdot \mathbf{x},$$

where

$$A = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{pmatrix}.$$

Therefore, the behaviour of the distribution of porridge after a while is controlled by the behavour of powers A^n.

It turns out that the eigenvalues of A are

$$1, \cos\frac{2\pi}{5} \approx 0.309, \cos\frac{4\pi}{5} \approx -0.809, \cos\frac{6\pi}{5} \approx -0.809, \cos\frac{8\pi}{5} \approx 0.309.$$

These are already quite hard to compute, since the equation $\det(A - cI_5) = 0$ is a degree 5 polynomial, for which no formulas for solution are available. It however is possible to solve it because $A = \frac{1}{2}(T + T^{-1})$, where T is the transformation that moves the contents of the bowls one seat clockwise. (Since A replaces the content of each bowl by the average of its clockwise and its counterclockwise neighbour). For the matrix T, we have $\det(T - cI_5) = 1 - c^5$, so T has distinct (complex) eigenvalues

$$1, \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5},$$

and $\frac{1}{2}(T+T^{-1})$ has the eigenvalues equal to $\frac{1}{2}(c+c^{-1})$, where c is an eigenvalue of T, so it remains to notice that $\frac{1}{2}(\cos\alpha + i\sin\alpha + (\cos\alpha + i\sin\alpha)^{-1}) = \cos\alpha$.

Some of the eigenvalues of A are equal to each other, which potentially can bring trouble (recall the example of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$), but in fact in this case there is a basis of eigenvectors. This happens because the eigenvalues of T are all distinct, so T has a basis of eigenvectors, and hence A does.

Let us denote choose a basis of eigenvectors of A, which we call $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$. Relative to this basis, the matrix A is transformed, using the respective transition matrix, into the matrix

$$B \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.309 & 0 & 0 & 0 \\ 0 & 0 & -0.809 & 0 & 0 \\ 0 & 0 & 0 & -0.809 & 0 \\ 0 & 0 & 0 & 0 & 0.309 \end{pmatrix},$$

and the matrix A^{100} transforms to

$$B^{100} \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 10^{-51} & 0 & 0 & 0 \\ 0 & 0 & 6 \cdot 10^{-10} & 0 & 0 \\ 0 & 0 & 0 & 6 \cdot 10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 10^{-51} \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This latter matrix sends a vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5$ to $c_1\mathbf{v}_1$, where $\mathbf{v}_1$ is the eigenvector corresponding to the eigenvalue 1. This vector is very easy to guess: all its coordinates are equal to each other (then averaging does not change it). This shows that after many iterations, each vector is sent to (approximately) a vector with equal coordinates, so the distribution of porridge is closer and closer to equal.

Remark 1. For 4, 6, or indeed any even number of kids, the conclusion would not work, since the corresponding matrix also has the eigenvalue -1 (hence the oscillation in the example we considered).