MA 1111/1212: Linear Algebra
Tutorial problems, November 25, 2015

1. (**a**) Yes. It is closed under taking sums and computing scalar multiples, and contains zero and opposite vectors by inspection, e.g. if $f(1) = 0$ and $g(1) = 0$, then $(f+g)(1) = f(1) + g(1) = 0$. All required properties follow from the same properties in the space of all polynomials.

Another proof: if a polynomial $f(x)$ has 1 as a root, then $f(x) = (x-1)f_1(x)$ for some polynomial $f_1(x)$. Indeed, we can do long division of $f(x)$ by $(x-1)$, and the remainder is a polynomial of degree 0, so a constant c, $f(x) = (x-1)q(x)+c$. Substituting $x = 1$, we get $c = f(1) = 0$. Therefore, $f(x)+g(x) = (x-1)f_1(x)+(x-1)g_1(x) = (x-1)(f_1(x)+g_1(x)) = 0$.

(**b**) Yes. It is closed under taking sums and computing scalar multiples, and contains zero and opposite vectors by inspection, e.g. if $f(1) = f(2) = 0$ and $g(1) = g(2) = 0$, then $(f+g)(1) = f(1) + g(1) = 0 = f(2) + g(2) = (f+g)(2)$. All properties follow from the same properties in the space of all polynomials.

(**c**) No. For example, both $x-1$ and $x-2$ belong to that subset, but their sum $x-1+x-2 = 2x-3$ does not.

2. Let us take a generic polynomial of degree less than 100, $f(x) = a_{99}x^{99}+\cdots+a_1x+a_0$.

In the first case, the condition $f(1) = 0$ is one linear equation on the coefficients of the polynomial, $a_{99} + a_{98} + \cdots + a_1 + a_0 = 0$. Clearly, a_{99} is the pivotal variable, and others are free variables. Exhibiting a basis of the solution space of this equation, we obtain a basis $1-x^{99}$, $x-x^{99}$, …, $x^{98}-x^{99}$ of the vector space in question. Therefore, the dimension is 99.

(Another idea: $f(x) = (x-1)f_1(x)$, where $f_1(x)$ is the space of polynomials of degree 98, so the polynomials $(x-1)$, $(x-1)x$, $(x-1)x^2$, …, $(x-1)x^{98}$ form a basis).

In the second case, the conditions $f(1) = f(2) = 0$ give us the following linear equations: $a_{99} + a_{98} + \cdots + a_1 + a_0 = 0$, $2^{99}a_{99} + 2^{98}a_{98} + \cdots + 2a_1 + a_0 = 0$. By direct inspection, the reduced row echelon form of the corresponding matrix has two pivots, so there are 98 free variables, and the dimension is 98.

3. Solving the system of equations $c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$, we get $c_1 = -1/2$, $c_2 = 3/2$, $c_3 = 7/2$.

4. Since $\dim P_2 = 3$, it is enough to show that these polynomials are linearly independent (if there is a polynomial g not equal to a linear combination of these three, we would have four linearly independent vectors in a 3-dimensional space, a contradiction). If $c_1(1+t^2)+c_2(2-t+t^2)+c_3(t-t^2) = 0$, then $c_1+2c_2 = 0$, $-c_2+c_3 = 0$, and $c_1+c_2-c_3 = 0$. From these equations, we have $c_1 = -2c_2$, $c_2 = c_3$, $0 = c_1+c_2-c_3 = -2c_2+c_2-c_2 = -2c_2$, which implies $c_1 = c_2 = c_3 = 0$.

The condition $c_1(1+t^2) + c_2(2-t+t^2) + c_3(t-t^2) = t^2+4t+4$ implies $c_1 + 2c_2 = 4$, $-c_2 + c_3 = 4$, and $c_1 + c_2 - c_3 = 1$, so $c_1 = 4-2c_2$, $c_3 = c_2+4$, $1 = c_1 + c_2 - c_3 = 4 - 2c_2 + c_2 - c_2 - 4 = -2c_2$, so $c_2 = -1/2$, $c_1 = 5$, $c_3 = 7/2$.