MA 1111/1212: Linear Algebra
Tutorial problems, December 9, 2015

1. Let us compute the images of the basis vectors: $1 \mapsto (3-7i)\cdot 1 = 3-7i$), $i \mapsto (3-7i)\cdot i = 7+3i$. This instantly leads to the matrix $\begin{pmatrix} 3 & 7 \\ -7 & 3 \end{pmatrix}$.

2. Let us compute the images of the basis vectors:

$$e_1 \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = -e_1 - e_2 + e_3,$$

$$e_2 \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -e_1 + e_4,$$

$$e_3 \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e_1 - e_4,$$

$$e_4 \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = e_2 - e_3 + e_4.$$

This immediately leads to the matrix

$$\begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

3. (**a**) Let us compute the transition matrix $M_{\mathbf{e},\mathbf{f}}$. Solving the systems of equations $f_1 = c_{11}e_1 + c_{21}e_2$ and $f_2 = c_{12}e_1 + c_{22}e_2$, we get $c_{11} = 7$, $c_{21} = -3$, $c_{12} = 30$, $c_{22} = -13$, therefore $M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 7 & 30 \\ -3 & -13 \end{pmatrix}$. Therefore, we have

$$A_{\varphi,\mathbf{f}} = M_{\mathbf{e},\mathbf{f}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 171 & 731 \\ -40 & -171 \end{pmatrix}.$$

(**b**) We clearly have $M_{\mathbf{v},\mathbf{e}} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, therefore $M_{\mathbf{e},\mathbf{v}} = M_{\mathbf{v},\mathbf{e}}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$. Therefore,

$$A_{\varphi,\mathbf{v}} = M_{\mathbf{e},\mathbf{v}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{v}} = \begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}.$$

4. We have $\det(A - aI) = (a-3)(a-2) - 2 = a^2 - 5a + 4 = (a-1)(a-4)$. This means that we should expect the linear map given by this matrix to have, relative to some basis, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. To find the corresponding basis, we solve the equations $Ax = x$ and $Ax = 4x$. Solving these, we find solutions $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. Thus, if we put $C = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$, we have $C^{-1}AC = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, and $C^{-1}A^nC = \begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix}$. Therefore,

$$A^n = C\begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix}C^{-1} = \begin{pmatrix} \frac{2\cdot 4^n+1}{3} & \frac{2\cdot 4^n-2}{3} \\ \frac{4^n-1}{3} & \frac{4^n+2}{3} \end{pmatrix}.$$