1111: Linear Algebra I

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Lecture 12

Previously on...

Theorem. Suppose that $\det(A) \neq 0$. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Proof. Indeed, take the formula $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I_n$, and divide by $\det(A)$. $\square$

This theorem shows that not only a matrix is invertible when the determinant is not equal to zero, but also that you can compute the inverse by doing exactly one division; all other operations are addition, subtraction, and multiplication.

Cramer's formula for systems of linear equations

We know that if A is invertible then $Ax = b$ has just one solution $x = A^{-1}b$. Let us plug in the formula for A^{-1} that we have:

$$x = A^{-1}b = \frac{1}{\det(A)}\,\mathrm{adj}(A)b\ .$$

When we compute $\mathrm{adj}(A)b = C^T b$, we get the vector whose k-th entry is

$$C^{1k}b_1 + C^{2k}b_2 + \ldots + C^{nk}b_n\ .$$

What does it look like? It looks like a k-th column expansion of some determinant, more precisely, of the determinant of the matrix A_k which is obtained from A by replacing its k-th column with b. (This way, the cofactors of that column do not change).

Theorem. (Cramer's formula) Suppose that $\det(A) \neq 0$. Then coordinates of the only solution to the system of equations $Ax = b$ are

$$x_k = \frac{\det(A_k)}{\det(A)}\ .$$

Summary of systems of linear equations

Theorem. Let us consider a system of linear equations $Ax = b$ with n equations and n unknowns. The following statements are equivalent:

(a) the *homogeneous* system $Ax = 0$ has only the trivial solution $x = 0$;

(b) the reduced row echelon form of A is I_n;

(c) $\det(A) \neq 0$.

(d) the matrix A is invertible;

(e) the system $Ax = b$ has exactly one solution;

Proof. In principle, to show that five statements are equivalent, we need to do a lot of work. We could, for each pair, prove that they are equivalent, altogether $5 \cdot 4 = 20$ proofs. We could prove that $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$, altogether 8 proofs. What we shall do instead is prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$, just 5 proofs.

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(*d*) the matrix A is invertible;

(*e*) the system $Ax = b$ has exactly one solution;

Proof. $(a) \Rightarrow (b)$: by contradiction, if the reduced row echelon form has a row of zeros, we get free variables.
$(b) \Rightarrow (c)$: follows from properties of determinants, elementary operations multiply the determinant by nonzero scalars.
$(c) \Rightarrow (d)$: proved in several different ways already.
$(d) \Rightarrow (e)$: discussed early on, if A is invertible, then $x = A^{-1}b$ is clearly the only solution to $Ax = b$.
$(e) \Rightarrow (a)$: by contradiction, if v a solution to $Ax = b$ and w is a nontrivial solution to $Ay = 0$, then $v + w$ is another solution to $Ax = b$.

Summary of systems of linear equations

A very important consequence (finite dimensional Fredholm alternative):

> *For an $n \times n$-matrix A, the system $Ax = b$ either has exactly one solution for every b, or has infinitely many solutions for some choices of b and no solutions for some other choices.*

In particular, to prove that $Ax = b$ *has* solutions for every b, it is enough to prove that $Ax = 0$ has only the trivial solution.

An example for the Fredholm alternative

Let us consider the following question:

> *Given some numbers in the first row, the last row, the first column, and the last column of an $n \times n$-matrix, is it possible to fill the numbers in all the remaining slots in a way that each of them is the average of its 4 neighbours?*

This is the "discrete Dirichlet problem", a finite grid approximation to many foundational questions of mathematical physics.

An example for the Fredholm alternative

For instance, for $n = 4$ we may face the following problem: find a, b, c, d to put in the matrix

$$\begin{pmatrix} 4 & 3 & 0 & 1.5 \\ 1 & a & b & -1 \\ 0.5 & c & d & 2 \\ 2.1 & 4 & 2 & 1 \end{pmatrix}$$

so that

$$\begin{cases} a = \frac{1}{4}(3 + 1 + b + c), \\ b = \frac{1}{4}(a + 0 - 1 + d), \\ c = \frac{1}{4}(a + 0.5 + d + 4), \\ d = \frac{1}{4}(b + c + 2 + 2). \end{cases}$$

This is a system with 4 equations and 4 unknowns.

An example for the Fredholm alternative

In general, we shall be dealing with a system with $(n-2)^2$ equations and $(n-2)^2$ unknowns.

Note that according to the Fredholm alternative, it is enough to prove that for the zero boundary data we get just the trivial solution. Let a_{ij} be a solution for the zero boundary data. Let a_{PQ} be the largest element among them. Since

$$a_{PQ} = \frac{1}{4}(a_{P-1,Q}+a_{P,Q-1}+a_{P+1,Q}+a_{P,Q+1}) \leq \frac{1}{4}(a_{PQ}+a_{PQ}+a_{PQ}+a_{PQ}),$$

the neighbours of a_{PQ} must all be equal to a_{PQ}. Similarly, their neighbours must be equal to a_{PQ} etc., and it propagates all the way to the boundary, so we observe that $a_{PQ}=0$. The same argument appliest with the smallest element, and we conclude that all elements must be equal to zero. This, as we already realised, proves that for every choice of the boundary data the solution is unique.

Summary of systems of linear equations

For systems of equations where the number of equations is not necessarily equal to the number of unknowns, there is one key result that we shall use extensively in the further parts of the module.

> *A homogeneous system $Ax = 0$ with n unknowns and $m < n$ equations always has a nontrivial solution.*

The proof is completely trivial. Indeed, there will be no inconsistencies of the type $0 = 1$, and there will be at least one free unknown since $m < n$.

An application of determinants: Vandermonde determinant

Let $x_1, \ldots, x_n$ be scalars. The *Vandermonde determinant* $V(x_1, \ldots, x_n)$ is the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}.$$

Theorem. We have

$$V(x_1, \ldots, x_n) = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1) \cdots (x_n - x_{n-1}) = \prod_{i<j} (x_j - x_i).$$

The Vandermonde determinant

A sneaky "proof": we see that $V(x_1, \ldots, x_n) = 0$ whenever $x_i = x_j$ for $i \neq j$ (two equal columns). Therefore, the polynomial expression $V(x_1, \ldots, x_n)$ is divisible by all $x_i - x_j$ for $i > j$. But the degree of $V(x_1, \ldots, x_n)$ is $1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}$ (because we take one element from each row), and the degree of the product

$$(x_2 - x_1)(x_3 - x_2)(x_3 - x_1) \cdots (x_n - x_{n-1})$$

is $1 + 2 + \cdots + n - 1$, so these polynomial expression differ by a scalar multiple. Comparing the coefficients of $x_2 x_3^2 \cdots x_n^{n-1}$ (the diagonal term), we find that both coefficients are 1, so there is an equality. $\square$

There are some "gaps" that are not hard to fill but need to be filled. Those who take the module 2214 next year, will be able to complete the proof formally, others need some trust.