1111: Linear Algebra I

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Lecture 13

Previously on...

Let $x_1, \ldots, x_n$ be scalars. The *Vandermonde determinant* $V(x_1, \ldots, x_n)$ is the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \ldots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \ldots & x_n^{n-1} \end{pmatrix} .$$

Theorem. We have

$$V(x_1, \ldots, x_n) = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1) \cdots (x_n - x_{n-1}) = \prod_{i<j} (x_j - x_i).$$

The Vandermonde determinant

Theorem. We have

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Proof: Let us subtract, for each $i = n-1, n-2, \ldots, 1$, the row i times x_1 from the row $i+1$. Combining rows does not change the determinant, so we conclude that $V(x_1,\ldots,x_n)$ is equal to the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & x_2-x_1 & x_3-x_1 & \ldots & x_n-x_1 \\ 0 & x_2^2-x_1x_2 & x_3^2-x_1x_3 & \ldots & x_n^2-x_1x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-1}-x_1x_2^{n-2} & x_3^{n-1}-x_1x_3^{n-2} & \ldots & x_n^{n-1}-x_1x_n^{n-2} \end{pmatrix}.$$

The Vandermonde determinant

Let us expand the determinant

$$\det\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 0 & x_2-x_1 & x_3-x_1 & \dots & x_n-x_1\\ 0 & x_2^2-x_1x_2 & x_3^2-x_1x_3 & \dots & x_n^2-x_1x_n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & x_2^{n-1}-x_1x_2^{n-2} & x_3^{n-1}-x_1x_3^{n-2} & \dots & x_n^{n-1}-x_1x_n^{n-2}\end{pmatrix}$$

along the first column, the result is

$$\det\begin{pmatrix}x_2-x_1 & x_3-x_1 & \dots & x_n-x_1\\ x_2^2-x_1x_2 & x_3^2-x_1x_3 & \dots & x_n^2-x_1x_n\\ \vdots & \vdots & \ddots & \vdots\\ x_2^{n-1}-x_1x_2^{n-2} & x_3^{n-1}-x_1x_3^{n-2} & \dots & x_n^{n-1}-x_1x_n^{n-2}\end{pmatrix}.$$

The Vandermonde determinant

We note that the k-th column of the determinant

$$\det\begin{pmatrix} x_2 - x_1 & x_3 - x_1 & \ldots & x_n - x_1 \\ x_2^2 - x_1x_2 & x_3^2 - x_1x_3 & \ldots & x_n^2 - x_1x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-1} - x_1x_2^{n-2} & x_3^{n-1} - x_1x_3^{n-2} & \ldots & x_n^{n-1} - x_1x_n^{n-2} \end{pmatrix}$$

is divisible by $x_{k+1} - x_1$, so it is equal to

$$(x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)\det\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_2 & x_3 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2} & x_3^{n-2} & \ldots & x_n^{n-2} \end{pmatrix},$$

so we encounter a smaller Vandermonde determinant, and can proceed by induction. □

The Vandermonde determinant

An important consequence: the Vandermonde determinant is not equal to zero if and only if $x_1, \dots, x_n$ are all distinct.

Theorem. For each n distinct numbers $x_1, \dots, x_n$, and each choice of $a_1, \dots, a_n$, there exists a unique polynomial $f(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$ of degree at most $n-1$ such that $f(x_1) = a_1, \dots, f(x_n) = a_n$.

Proof: Let us figure out what conditions are imposed on the coefficients $c_0, \dots, c_{n-1}$:

$$\begin{cases} c_0 + c_1x_1 + \cdots + c_{n-1}x_1^{n-1} = a_1, \\ c_0 + c_1x_2 + \cdots + c_{n-1}x_2^{n-1} = a_2, \\ \dots, \\ c_0 + c_1x_n + \cdots + c_{n-1}x_n^{n-1} = a_n \ . \end{cases}$$

The matrix of this system is the transpose of the Vandermonde matrix!

The Vandermonde determinant

We conclude that the conditions we wish to observe are of the form

$Ax = b$, where $b = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $\det(A) = V(x_1 \ldots, x_n)$. Since $x_1, \ldots, x_n$ are distinct, $\det(A) \neq 0$, and the system has exactly one solution for any choice of the vector b. $\square$

Remark. In fact, one can write the formula for $f(x)$ directly. The following neat formula for $f(x)$ is called the *Lagrange interpolation formula*:

$$f(x) = \sum_{i=1}^{n} a_i \frac{(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} .$$

The conditions $f(x_i) = a_i$ are easily checked by inspection.

The coordinate vector space $\mathbb{R}^n$

We already used vectors in n dimensions when talking about systems of linear equations. However, we shall now introduce some further notions and see how those notions may be applied.

Recall that the coordinate vector space $\mathbb{R}^n$ consists of all columns of height n with real entries, which we refer to as *vectors*.

Let $v_1, \ldots, v_k$ be vectors, and let $c_1, \ldots, c_k$ be real numbers. The *linear combination* of vectors $v_1, \ldots, v_k$ with coefficients $c_1, \ldots, c_k$ is, quite unsurprisingly, the vector $c_1 v_1 + \cdots + c_k v_k$.

The vectors $v_1, \ldots, v_k$ are said to be *linearly independent* if the only linear combination of this vector which is equal to the zero vector is the combination where all coefficients are equal to 0. Otherwise those vectors are said to be *linearly dependent*.

The vectors $v_1, \ldots, v_k$ are said to *span* $\mathbb{R}^n$, or to form a *complete set of vectors*, if every vector can be written as some linear combination of those vectors.

Linear independence and span: examples

The vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}3\\0\end{pmatrix}$ are linearly dependent:

$$(-3)\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}3\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

The vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ are linearly independent: if $c_1\begin{pmatrix}1\\0\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, we have $c_1+c_2=0$ and $c_2=0$, so both coefficients in the linear combination are zero.

Linear independence and span: examples

The vector $\begin{pmatrix}1\\1\end{pmatrix}$ does not span $\mathbb{R}^2$: any linear combination in this case is just a scalar multiple, so we can only get vectors with equal coordinates.

The vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ span $\mathbb{R}^2$: if we look, given a_1, a_2, for c_1, c_2 such that $c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$, we have $c_1 + c_2 = a_1$ and $c_2 = a_2$, which can be easily solved for c_1, c_2, so any vector can be written as a combination of these.

Linear independence and span: examples

If a system of vectors contains the zero vector, these vectors may not be linearly independent, since it is enough to take the zero vector with a nonzero coefficient.

If a system of vectors contains two equal vectors, or two proportional vectors, these vectors may not be linearly independent. More generally, several vectors are linearly dependent if and only if one of those vectors can be represented as a linear combination of others. (*Exercise:* prove that last statement).

The standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent; they also span $\mathbb{R}^n$.

If the given vectors are linearly independent, then removing some of them keeps them linearly independent. If the given vectors span $\mathbb{R}^n$, then throwing in some extra vectors does not destroy this property.

Linear combinations and systems of linear equations

Let us make one very important observation:

For an $n \times k$-matrix A and a vector x of height k, the product Ax is the linear combination of columns of A whose coefficients are the coordinates of the vector x. If $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$, and $A = (v_1 \mid v_2 \mid \cdots \mid v_k)$, then $Ax = x_1 v_1 + \cdots + x_k v_k$.

We already utilised that when working with systems of linear equations.

Linear independence and span

Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. Consider the $n \times k$-matrix A whose columns are these vectors.

Clearly, the vectors $v_1, \ldots, v_k$ are linearly independent if and only if the system of equations $Ax = 0$ has only the trivial solution. This happens if and only if there are no free variables, so the reduced row echelon form of A has a pivot in every column.

Clearly, the vectors $v_1, \ldots, v_k$ span $\mathbb{R}^n$ if and only if the system of equations $Ax = b$ has solutions for every b. This happens if and only if the reduced row echelon form of A has a pivot in every row. (Indeed, otherwise for some b we shall have the equation $0 = 1$).

In particular, if $v_1, \ldots, v_k$ are linearly independent in $\mathbb{R}^n$, then $k \leq n$ (there is a pivot in every column of A, and at most one pivot in every row), and if $v_1, \ldots, v_k$ span $\mathbb{R}^n$, then $k \geq n$ (there is a pivot in every row, and at most one pivot in every column).

Bases of $\mathbb{R}^n$

We say that vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ form a *basis* if they are linearly independent and they span $\mathbb{R}^n$.

Theorem. Every basis of $\mathbb{R}^n$ consists of exactly n elements.

Proof. We know that if $v_1, \ldots, v_k$ are linearly independent, then $k \leq n$, and if $v_1, \ldots, v_k$ span $\mathbb{R}^n$, then $k \geq n$. Since both properties are satisfied, we must have $k = n$.

Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$. Consider the $n \times n$-matrix A whose columns are these vectors. Our previous results immediately show that $v_1, \ldots, v_n$ form a basis if and only if the matrix A is invertible (for which we had many equivalent conditions earlier).