1111: Linear Algebra I

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Lecture 19

Let us start with an example of computing coordinates. Let $V = \mathbb{R}^2$. We take the basis $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Let us find find the coordinates of the vector $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ in $\mathbb{R}^2$ relative to the basis e_1, e_2. Recall that $c_1e_1+c_2e_2 = (e_1 \mid e_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Therefore, coordinates of $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ relative to this basis are $(e_1 \mid e_2)^{-1}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 16 \\ -3 \end{pmatrix}$.

Change of coordinates

Before we proceed with more theory, let us also discuss an example of computing a transition matrix, in the same fashion as we just computed coordinates. Let $V = \mathbb{R}^2$. We shall compute the transition matrix $M_{\mathbf{e},\mathbf{f}}$ from the basis $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ to the basis $f_1 = \begin{pmatrix} 13 \\ -12 \end{pmatrix}, f_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. We note that $(e_1 \mid e_2)M_{ef} = (f_1 \mid f_2)$, so $M_{ef} = (e_1 \mid e_2)^{-1}(f_1 \mid f_2) = \begin{pmatrix} -87 & -7 \\ 25 & 2 \end{pmatrix}$.

If $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be two different bases of V, we can compute coordinates of each vector v with respect to either of those bases, so that

$$v = x_1e_1 + \cdots + x_ne_n$$

and

$$v = y_1f_1 + \cdots + y_nf_n.$$

Our goal now is to figure out how these are related. Let us denote $v_{\mathbf{e}}$ the column of coordinates of v relative to the first basis, and by $v_{\mathbf{f}}$ the column of coordinates of v relative to the second basis.

Lemma 1. *We have*

$$v_{\mathbf{e}} = M_{\mathbf{e},\mathbf{f}}v_{\mathbf{f}}.$$

In plain words, if we call $e_1, \ldots, e_n$ the "old basis" and $f_1, \ldots, f_n$ the "new basis", then this system tells us that the product of the transition matrix with the columns of new coordinates of a vector is equal to the column of old coordinates.

Proof. The proof is fairly straightforward: we take the formula

$$v = y_1f_1 + \cdots + y_nf_n,$$

and substitute instead of f_i's their expressions in terms of e_j's:

$$\begin{aligned} f_1 &= a_{11}e_1 + a_{21}e_2 + \cdots + a_{m1}e_m, \\ f_2 &= a_{12}e_1 + a_{22}e_2 + \cdots + a_{m2}e_m, \\ &\ldots \\ f_n &= a_{1n}e_1 + a_{2n}e_2 + \cdots + a_{mn}e_m. \end{aligned}$$

What we get is

$$y_1(a_{11}e_1+a_{21}e_2+\cdots+a_{n1}e_n)+y_2(a_{12}e_1+a_{22}e_2+\cdots+a_{n2}e_n)+\ldots+y_n(a_{1n}e_1+a_{2n}e_2+\cdots+a_{nn}e_n)=$$
$$=(a_{11}y_1+a_{12}y_2+\cdots+a_{1n}y_n)e_1+\cdots+(a_{n1}y_1+a_{n2}y_2+\cdots+a_{nn}y_n)e_n.$$

Since we know that coordinates are uniquely defined, we conclude that

$$a_{11}y_1+a_{12}y_2+\cdots+a_{1n}y_n=x_1,$$
$$\ldots$$
$$a_{n1}y_1+a_{n2}y_2+\cdots+a_{nn}y_n=x_n,$$

which is what we want to prove. □

Let us remark that there is a slightly confusing aspect of transition matrices that needs to be noted. Originally, the transition matrix was defined as a matrix of coefficients expressing the "new" basis via the "old" basis. Now we just proved that it also expresses the "old" coordinates via the "new" coordinates. Thus, bases and coordinates transform in opposite ways. This later on gives rise to the notions of *covariance* and *contravariance* in theoretical physics and differential geometry. Covariant objects transform like vectors, and contravariant objects transform like coordinates.

We shall now prove a useful "multiplicative" property of transition matrices.

Lemma 2. *We have (for three different bases* $e_1,\ldots,e_n$, $f_1,\ldots,f_n$, $g_1,\ldots,g_n$*)*

$$M_{\mathbf{e},\mathbf{f}}M_{\mathbf{f},\mathbf{g}}=M_{\mathbf{e},\mathbf{g}}$$

and

$$M_{\mathbf{e},\mathbf{f}}M_{\mathbf{f},\mathbf{e}}=I_n.$$

Proof. Applying the formula above twice, we have

$$v_{\mathbf{e}}=M_{\mathbf{e},\mathbf{f}}v_{\mathbf{f}}=M_{\mathbf{e},\mathbf{f}}M_{\mathbf{f},\mathbf{g}}v_{\mathbf{g}}.$$

But we also have

$$v_{\mathbf{e}}=M_{\mathbf{e},\mathbf{g}}v_{\mathbf{g}}.$$

Therefore

$$M_{\mathbf{e},\mathbf{f}}M_{\mathbf{f},\mathbf{g}}v_{\mathbf{g}}=M_{\mathbf{e},\mathbf{g}}v_{\mathbf{g}}$$

for every $v_{\mathbf{g}}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so $M_{\mathbf{e},\mathbf{f}}M_{\mathbf{f},\mathbf{g}}=M_{\mathbf{e},\mathbf{g}}$ as required. Since manifestly we have $M_{\mathbf{e},\mathbf{e}}=I_n$, we conclude by letting $g_k=e_k$, $k=1,\ldots,n$, that $M_{\mathbf{e},\mathbf{f}}M_{\mathbf{f},\mathbf{e}}=I_n$. □

Linear maps

Definition 1. Suppose that V and W are two vector spaces. A function $f\colon V\to W$ is said to be a *linear map*, or a *linear transformation*, if

- for $v_1,v_2\in V$, we have $f(v_1+v_2)=f(v_1)+f(v_2)$,
- for $c\in\mathbb{R}$, $v\in V$, we have $f(c\cdot v)=c\cdot f(v)$.

When $V=W$, a linear map $f\colon V\to V$ is sometimes called a *linear operator*.

The notion of a linear map is one of the most important notions of linear algebra. So far in your calculus class you discussed functions of one variable and studied those functions using derivatives. Once you move to vector functions of several variables, an inevitable move for purposes of geometry and physics, derivatives of such functions are linear maps between vector spaces.

Lemma 3. *Suppose that* f *is a linear map. Then* $f(0) = 0$*, and* $f(-v) = -f(v)$.

Proof. This follows from $0 \cdot v = 0$ and $(-1) \cdot v = -v$. $\square$

Thus, we can say that a linear map is a function between vector spaces that preserves all the structures.

Example 1. Let us consider the vector spaces P_2 and P_3 of all polynomials in one variable x of degrees at most 2 and at most 3, respectively. We consider a map $X\colon P_2 \to P_3$ defined by $X(f(x)) = x \cdot f(x)$, and a map $D\colon P_3 \to P_2$ defined by $D(f(x)) = f'(x)$. These are easily checked to be linear maps.