1111: Linear Algebra I

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Lecture 21

Linear maps and change of coordinates

As the next step, let us study how matrices of linear maps transform under changes of coordinates.

Lemma 1. *Let $\varphi\colon V \to W$ be a linear map, and suppose that $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ are two bases of V, and $f_1, \ldots, f_m$ and $f_1', \ldots, f_m'$ are two bases of W. Then*

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'} = M_{\mathbf{f},\mathbf{f}'}^{-1} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}.$$

Proof. Let us take a vector $\mathbf{v} \in V$. On the one hand, the formula we proved last time tells us that

$$(\varphi(\mathbf{v}))_{\mathbf{f}'} = A_{\varphi,\mathbf{e}',\mathbf{f}'} \mathbf{v}_{\mathbf{e}'}.$$

On the other hand, applying various results we proved earlier, we have

$$(\varphi(\mathbf{v}))_{\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}}(\varphi(\mathbf{v})_{\mathbf{f}}) = M_{\mathbf{f}',\mathbf{f}}(A_{\varphi,\mathbf{e},\mathbf{f}} \mathbf{v}_{\mathbf{e}}) = M_{\mathbf{f}',\mathbf{f}}(A_{\varphi,\mathbf{e},\mathbf{f}}(M_{\mathbf{e},\mathbf{e}'} \mathbf{v}_{\mathbf{e}'})) = (M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}) \mathbf{v}_{\mathbf{e}'}.$$

Therefore,

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} \mathbf{v}_{\mathbf{e}'} = (M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}) \mathbf{v}_{\mathbf{e}'}$$

for every $\mathbf{v}_{\mathbf{e}'}$. From our previous classes we know that knowing $A\mathbf{v}$ for all vectors $\mathbf{v}$ completely determines the matrix A, so

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = (M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}) = (M_{\mathbf{f},\mathbf{f}'}^{-1} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'})$$

because of properties of transition matrices proved earlier. ∎

Let us remarks that the formula

$$A_{\varphi,\mathbf{e}',\mathbf{f}'} = M_{\mathbf{f}',\mathbf{f}} A_{\varphi,\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{e}'}.$$

shows that changing from the coordinate systems $\mathbf{e}, \mathbf{f}$ to *some* other coordinate system amounts to multiplying the matrix $A_{\varphi,\mathbf{e},\mathbf{f}}$ by some invertible matrices on the left and on the right, so effectively it is nothing but performing a certain number of elementary row and column operations on this matrix.

Linear operators have the same vector space for arguments and values, so it only really makes sense to use the same coordinate system for the input and the output. By definition, the matrix of a linear operator $\varphi\colon V \to V$ relative to the basis $e_1, \ldots, e_n$ is

$$A_{\varphi,\mathbf{e}} := A_{\varphi,\mathbf{e},\mathbf{e}}.$$

Lemma 2. *For a linear operator $\varphi\colon V \to V$, and two bases $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ of V, we have*

$$A_{\varphi,\mathbf{e}'} = M_{\mathbf{e},\mathbf{e}'}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{e}'}.$$

Proof. This is a particular case of the previous result. ∎

Therefore, unlike matrices of linear maps between different spaces, matrices of linear operators cannot be transformed by arbitrary choices of row/column operations: when performing some operation on rows, one has to simultaneously perform the inverse operation on columns!

In general, one important goal of linear algebra is to find simple expressions for various structures. For instance, if V is a finite-dimensional vector space (no extra structure), then choosing a basis identifies it with $\mathbb{R}^n$ for $n = \dim(V)$. If $\varphi\colon V \to W$ is a linear map between finite-dimensional vector spaces, we can also simplify it very significantly. As we saw before, once we choose a basis in V and a basis in W, a linear map is represented by a matrix, and changing bases amounts to performing row and column operations. As a consequence, we can first bring a matrix to its reduced row echelon form by row operations, and then move columns with pivots to the beginning and use them to cancel all elements in non-pivotal columns. Therefore, the number of pivots is the only geometric invariant of a linear map. We shall discuss it in more detail in the beginning of the second semester.

The case of a linear operator $\varphi\colon V \to V$ is more intricate. We established that for a square matrix A, the change $A \mapsto C^{-1}AC$ with an invertible matrix C, corresponds to the situation where A is viewed as a matrix of a linear operator, and C is viewed as a transition matrix for a coordinate change. You verified in your earlier home assignments that $\operatorname{tr}(C^{-1}AC) = \operatorname{tr}(A)$ and $\det(C^{-1}AC) = \det(A)$; these properties imply that the trace and the determinant do not depend on the choice of coordinates, and hence reflect some "geometric" properties of a linear transformation.

In case of the determinant, those properties have been hinted at in our previous classes: determinants compute how a linear transformation changes volumes of solids. In the case of the trace, the situation is a bit more subtle: the best one can get is a formula

$$\det(I_n + \varepsilon A) \approx 1 + \varepsilon \operatorname{tr}(A),$$

where $\approx$ means that the correction term is a polynomial expression in ε of magnitude bounded by a constant multiple of ε^2 (for small ε).

Example of change of coordinates

Example 1. Let us take two bases of $\mathbb{R}^2$: $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $e_1' = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $e_2' = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Suppose that the matrix of a linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ relative to the first basis is $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. Let us compute its matrix relative to the second basis. For that, we first compute the transition matrix $M_{\mathbf{e},\mathbf{e}'}$. We have

$$M_{\mathbf{e},\mathbf{e}'} = (e_1 \mid e_2)^{-1}(e_1' \mid e_2') = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix},$$

and

$$M_{\mathbf{e},\mathbf{e}'}^{-1} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}.$$

Therefore

$$A_{\varphi,\mathbf{e}'} = M_{\mathbf{e},\mathbf{e}'}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{e}'} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 10 & 9 \end{pmatrix}.$$

Observe that the trace and the determinant indeed have not changed.

In this example, one of the two matrices is much simpler than the other ones: the first matrix is diagonal, which simplifies all sorts of computations.