# 1111: Linear Algebra I

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Lecture 23

## Eigenvalues and eigenvectors

Last time we started discussed eigenvalues and eigenvectors. Recall that if $V$ is a vector space, $\varphi\colon V \to V$ is a linear operator, and $\mathbf{v} \neq 0$ a vector for which $\varphi(\mathbf{v})$ is proportional to $\mathbf{v}$, that is $\varphi(\mathbf{v}) = c \cdot \mathbf{v}$ for some $c$, then $\mathbf{v}$ is called an *eigenvector* of $\varphi$, and $c$ is the corresponding *eigenvalue*. If $\varphi$ is represented, relative to some basis, by an $n \times n$-matrix $A = A_{\varphi,\mathbf{e}}$, then eigenvalues are roots of the equation $\det(A - cI_n) = 0$. The left hand side of this equation is a degree $n$ polynomial in $c$. Then eigenvectors correspond to solutions to the system of equations $(A - c_0 I_n)x = 0$, where $c_0$ is an eigenvalue.

**Theorem 1.** *If all eigenvalues of $\varphi$ are distinct, then the corresponding eigenvectors form a basis of $V$.*

*Proof.* It is enough to show that they are linearly independent: if $\dim(V) = n$, then $n$ linearly independent vectors form a basis.

We shall prove that if $\mathbf{u}_1, \ldots, \mathbf{u}_k$, are eigenvectors of $\varphi$ with distinct eigenvalues, then they are linearly independent. Proof by induction on $k$; for $k = 1$ it is true because an eigenvector is a non-zero vector by definition. Assume that we proved it for $k$, and want to prove for $k+1$. Let us suppose that the eigenvectors $\mathbf{u}_1$, $\ldots$, $\mathbf{u}_{k+1}$ with distinct eigenvectors $c_1, \ldots, c_{k+1}$ are linearly dependent, so that $a_1\mathbf{u}_1 + \cdots + a_{k+1}\mathbf{u}_{k+1} = 0$. Applying the map $\varphi$, we get

$$0 = \varphi(0) = \varphi(a_1\mathbf{u}_1 + \cdots + a_{k+1}\mathbf{u}_{k+1}) = a_1\varphi(\mathbf{u}_1) + \cdots + a_{k+1}\varphi(\mathbf{u}_{k+1}) = a_1c_1\mathbf{u}_1 + \cdots + a_{k+1}c_{k+1}\mathbf{u}_{k+1}.$$

Subtracting from this $c_{k+1}(a_1\mathbf{u}_1 + \cdots + a_{k+1}\mathbf{u}_{k+1}) = c_{k+1} \cdot 0 = 0$, we get

$$a_1(c_1 - c_{k+1})\mathbf{u}_1 + \cdots + a_k(c_k - c_{k+1})\mathbf{u}_k = 0,$$

which is a linear dependence between $\mathbf{u}_1, \ldots, \mathbf{u}_k$. By induction hypothesis,

$$a_1(c_1 - c_{k+1}) = \cdots = a_k(c_k - c_{k+1}) = 0,$$

which, since the eigenvalues are distinct, means $a_1 = \cdots = a_k = 0$. Thus, our linear dependence becomes $0 \cdot \mathbf{u}_1 + \cdots + 0 \cdot \mathbf{u}_k + a_{k+1}\mathbf{u}_{k+1} = 0$, so $a_{k+1}\mathbf{u}_{k+1} = 0$, and thus $a_{k+1} = 0$, finishing the proof. □

If eigenvalues are not distinct, "anything can happen".

**Example 1.** Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We have $\det(A - cI_2) = (c-1)^2$, so the only eigenvalue is $1$. Moreover, all eigenvectors are scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so in this case there is no basis of eigenvectors.

**Example 2.** Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We have $\det(A - cI_2) = (1-c)^2$, so the only eigenvalue is $1$. We have $Ax = x$ for every vector $x$, so any basis is a basis of eigenvectors.

**Example 3.** Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have $\det(A - cI_2) = c^2 + 1$, so if we use real scalars, then there are no eigenvalues, and if scalars are complex, the eigenvalues are $\pm i$, and there is a basis consisting of eigenvectors.

In the next semester, we shall discuss this question in detail, and find "nearly diagonal" matrices representing arbitrary linear operators (the so called Jordan normal forms). We shall also prove that every symmetric matrix with real coefficients has a basis of eigenvectors, which is a very important result for analysis (symmetric matrices play the role of second derivatives).

## A toy example: "porridge problem"

Eigenvectors appear everywhere in applications of linear algebra. Google uses eigenvectors to decide which pages to show you first in results of a search. Facebook uses eigenvectors to guess who appears in your photos. Statistics, biology, economics, physics (especially quantum mechanics) etc. use eigenvectors to describe "pure" states in which a system can be. We shall now illustrate this philosophy using a simplistic toy problem.

> Five not quite hungry but quite playful kids are sat down around a round table. There is a bowl of porridge in front of each of them, and they generally have different amounts of porridge. They decide to play a game. Every minute each of them simultaneously gives one half of what they have to each of their neighbours. What is the dynamics of this process? Will they have roughly the same amounts in a while, or will the amounts of porridge in bowls keep changing significantly?

Intuitively, one would guess that after a while porridge will be distributed almost equally because of ongoing "averaging". However, one has to be careful. For example, if there are four kids, and amounts of porridge around the circle (in some units) are $1, 0, 1, 0$, then after one averaging this becomes $0, 1, 0, 1$, and after another averaging will be back to the original $1, 0, 1, 0$, so the picture will oscillate between these two distributions!