MA 1111: Linear Algebra I
Tutorial problems, November 22, 2018

1. Let us determine what this linear operator does to the basis vectors: $1 \mapsto (3-7i)\cdot 1 = 3-7i$, $i \mapsto (3-7i)\cdot i = 7+3i$. This instantly leads to the matrix $\begin{pmatrix} 3 & 7 \\ -7 & 3 \end{pmatrix}$.

2. (**a**) We have $M_{\mathbf{e},\mathbf{f}} = (e_1 \mid e_2)^{-1}(f_1 \mid f_2) = \begin{pmatrix} 7 & 30 \\ -3 & -13 \end{pmatrix}$. Therefore, we have

$$A_{\varphi,\mathbf{f}} = M_{\mathbf{e},\mathbf{f}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} 171 & 731 \\ -40 & -171 \end{pmatrix}.$$

(**b**) We clearly have $M_{\mathbf{v},\mathbf{e}} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, therefore $M_{\mathbf{e},\mathbf{v}} = M_{\mathbf{v},\mathbf{e}}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$. Therefore,

$$A_{\varphi,\mathbf{v}} = M_{\mathbf{e},\mathbf{v}}^{-1} A_{\varphi,\mathbf{e}} M_{\mathbf{e},\mathbf{v}} = \begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}.$$

3. We have $\det(A - cI_3) = -c^3 + 6c^2 - 12c + 8 = -(c-2)^3$. Therefore, all eigenvalues are equal to 2. Looking at the eigenvector condition $(A - 2I_3)x = 0$, we see that every eigenvector is proportional to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, so there is no basis of eigenvectors.

4. We have $\det(A - aI_2) = (a-3)(a-2) - 2 = a^2 - 5a + 4 = (a-1)(a-4)$. This means that we should expect the linear map given by this matrix to have, relative to some basis, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. To find the corresponding basis, we solve the equations $Ax = x$ and $Ax = 4x$. Solving these, we find solutions $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. Thus, if we put $C = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$, we have $C^{-1}AC = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, and $C^{-1}A^nC = \begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix}$. Therefore,

$$A^n = C \begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix} C^{-1} = \begin{pmatrix} \frac{2\cdot 4^n+1}{3} & \frac{2\cdot 4^n-2}{3} \\ \frac{4^n-1}{3} & \frac{4^n+2}{3} \end{pmatrix}.$$