MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for February 18, 2019

1. (**a**) The characteristic polynomial of A is $t^2 - t - 2 = (t-2)(t+1)$, so the eigenvalues are 2 and -1. We have $A - 2I = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$, and $A + I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$; both of these matrices have the rank equal to 1, the kernel of the first one is spanned by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and the kernel of the second one is spanned by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since our vector space is two-dimensional, each of these vectors can only give rise to a thread of length 1, and they form a basis. The Jordan normal form of our matrix is $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

(**b**) The characteristic polynomial of A is $t^2 - 2t + 1 = (t-1)^2$, so the only eigenvalue is 1. We have $A - I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$, and this matrix evidently is of rank 1. Also, $(A-I)^2 = 0$, so there is a stabilising sequence of subspaces $\operatorname{Ker}(A-I) \subset \operatorname{Ker}(A-I)^2 = V$. The dimension gap between these is equal to 1, and we have to find a relative basis. The kernel of $A - I$ is spanned by the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and for the relative basis we can take the vector $f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which makes up for the missing pivot. This vector gives rise to a thread $f, (A-I)f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the Jordan normal form $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

2. (**a**) The characteristic polynomial of A is $8 - 12t + 6t^2 - t^3 = (2-t)^3$, so the only eigenvalue is 2. We have $B = A - 2I = \begin{pmatrix} 2 & 9 & -5 \\ -4 & -10 & 6 \\ -6 & -13 & 8 \end{pmatrix}$, $B^2 = \begin{pmatrix} -2 & -7 & 4 \\ -4 & -14 & 8 \\ -8 & -28 & 16 \end{pmatrix}$, $B^3 = 0$, so there is a stabilising sequence of subspaces $\operatorname{Ker} B \subset \operatorname{Ker} B^2 \subset \operatorname{Ker} B^3 = V$. The dimension gap between the latter two is equal to 1. Let us describe $\operatorname{Ker} B^2$. It consists of the vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $-2x - 7y + 4z = 0$, so the basis is formed by the vectors $\begin{pmatrix} -7/2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$; computing the reduced column echelon form of the corresponding matrix gives us the relative basis vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ making up for the missing pivot. We have a thread $f, Bf = \begin{pmatrix} -5 \\ 6 \\ 8 \end{pmatrix}, B^2 f = \begin{pmatrix} 4 \\ 8 \\ 16 \end{pmatrix}$; the Jordan normal form of our matrix is $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

3. The characteristic polynomial of A is $2 - 5t + 4t^2 - t^3 = (1-t)^2(2-t)$, so the eigenvalues are 1 and 2. Furthermore, $\operatorname{rk}(A-I) = 2$, $\operatorname{rk}((A-I)^2) = 1$, $\operatorname{rk}(A-2I) = 2$, $\operatorname{rk}(A-2I)^2 = 2$. Thus, the kernels of powers of $A - 2I$ stabilise instantly, so we should expect a thread of length 1 for the eigenvalue 2, whereas the kernels of powers of $A - I$ do not stabilise for at least two steps, so that would give a thread of length at least 2, hence a thread of length 2 because our space is 3-dimensional. To determine the basis of $\operatorname{Ker}(A-2I)$, we solve the system $(A-2I)v = 0$ and obtain a vector $f = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. To deal with the eigenvalue 1, we see that the kernel of $A - I$ is spanned by the vector $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, and the kernel of $(A - I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -2 & -7 & 4 \\ -4 & -14 & 8 \end{pmatrix}$ is spanned by the vectors $\begin{pmatrix} -7/2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. Reducing the latter vectors using the former one, we end up with the vector $e = \begin{pmatrix} 0 \\ 1 \\ 7/4 \end{pmatrix}$, which gives rise to a thread $e, (A-I)e = \begin{pmatrix} -1/4 \\ 1/2 \\ 3/4 \end{pmatrix}$. Overall, a Jordan basis is given by $f, e, (A-I)e$, and the Jordan normal form of our matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

has a block of size 2 with 1 on the diagonal, and a block of size 1 with 2 on the diagonal.

4. The characteristic polynomial of A is $t^4 - t^3$, so the eigenvalues are 0 and 1. For the eigenvalue 0, we have $A^2 = \begin{pmatrix} 28 & 35 & 18 & 41 \\ 30 & 40 & 20 & 45 \\ -116 & -150 & -76 & -172 \\ 6 & 8 & 4 & 9 \end{pmatrix}$, $\mathrm{rk}(A^2) = 2$, and $A^3 = \begin{pmatrix} -8 & 8 & 0 & -4 \\ -10 & 10 & 0 & -5 \\ 36 & -36 & 0 & 18 \\ -2 & 2 & 0 & -1 \end{pmatrix}$, $\mathrm{rk}(A^3) = 1$, so there is a stabilising sequence of subspaces $\mathrm{Ker}\, A \subset \mathrm{Ker}\, A^2 \subset \mathrm{Ker}\, A^3$. At this points the kernels should stabilize, since we also have an eigenvector with the eigenvalue 1, and our space is 4-dimensional. For the eigenvalue 0, we should find a basis of $\mathrm{Ker}\, A^3$ relative to $\mathrm{Ker}\, A^2$.The former space consists of all vectors $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ with $-2x + 2y - t = 0$ and therefore has a basis of vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, whereas the latter has a basis of two vectors $\begin{pmatrix} -2/7 \\ -2/7 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -13/14 \\ -3/7 \\ 0 \\ 1 \end{pmatrix}$. Reducing the first set of vectors modulo the second one, we end up with the vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, for which we have $Af = \begin{pmatrix} 13 \\ 14 \\ -52 \\ 2 \end{pmatrix}$, $A^2 f = \begin{pmatrix} 18 \\ 20 \\ -76 \\ 4 \end{pmatrix}$. It remains to find an eigenvector with the eigenvalue 1; for such an eigenvector we can take $e = \begin{pmatrix} 4 \\ 5 \\ -18 \\ 1 \end{pmatrix}$. The vectors $f, Af, A^2 f, e$ form a Jordan basis, and the Jordan normal form is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.