MA 1112: Linear Algebra II
Selected answers/solutions to the assignment for March 19, 2019

1. (**a**) For $x_1 = x_2$ and $y_1 = y_2$ we get the value $2x_1y_1$ which also assumes negative values, so it is not a scalar product.

(**b**) For $x_1 = x_2$ and $y_1 = y_2$ we get the value x_1^2 which is nonnegative but vanishes for a nonzero vector $\begin{pmatrix}0\\1\end{pmatrix}$, so it is not a scalar product.

(**c**) This formula is manifestly bilinear and symmetric, and for $x_1 = x_2$ and $y_1 = y_2$ we get $x_1^2 + 7y_1^2$ which implies positivity, so it is a scalar product.

(**d**) For $x_1 = x_2$ and $y_1 = y_2$ we obtain $x_1^2 + 2x_1y_1 + y_1^2 = (x_1 + y_1)^2$, and there are nonzero vectors for which this vanishes, so it is not a scalar product.

(**e**) It is not symmetric, so it is not a scalar product.

2. First we make this set into a set of orthogonal vectors. We put

$$e_1 = f_1 = \begin{pmatrix}2\\1\\0\end{pmatrix},$$

$$e_2 = f_2 - \frac{(e_1, f_2)}{(e_1, e_1)}e_1 = \begin{pmatrix}-4/5\\8/5\\1\end{pmatrix},$$

$$e_3 = f_3 - \frac{(e_1, f_3)}{(e_1, e_1)}e_1 - \frac{(e_2, f_3)}{(e_2, e_2)}e_2 = \begin{pmatrix}1/7\\-2/7\\4/7\end{pmatrix}.$$

To conclude, we normalise the vectors, obtaining the answer

$$\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \frac{1}{\sqrt{105}}\begin{pmatrix}-4\\8\\5\end{pmatrix}, \quad \frac{1}{\sqrt{21}}\begin{pmatrix}1\\-2\\4\end{pmatrix}.$$

3. We first orthogonalise these vectors, noting that $\int_{-1}^{1} f(t)\,dt$ is equal to 0 if $f(t)$ is an odd function (this shows that our computations are actually quite easy, because even powers of t are automatically orthogonal to odd powers):

$$e_1 = 1,$$

$$e_2 = t - \frac{(1,t)}{(1,1)}1 = t,$$

$$e_3 = t^2 - \frac{(1,t^2)}{(1,1)}1 - \frac{(t,t^2)}{(t,t)}t = t^2 - \frac{1}{3},$$

$$e_4 = t^3 - \frac{(1,t^3)}{(1,1)}1 - \frac{(t,t^3)}{(t,t)}t - \frac{(t^2-\frac{1}{3}, t^3)}{(t^2-\frac{1}{3}, t^2-\frac{1}{3})}(t^2 - \frac{1}{3}) = t^3 - \frac{3}{5},$$

$$e_5 = t^4 - \frac{(1,t^4)}{(1,1)}1 - \frac{(t,t^4)}{(t,t)}t - \frac{(t^2-\frac{1}{3}, t^4)}{(t^2-\frac{1}{3}, t^2-\frac{1}{3})}(t^2 - \frac{1}{3}) - \frac{(t^3-\frac{3}{5}, t^4)}{(t^3-\frac{3}{5}, t^3-\frac{3}{5})}(t^3 - \frac{3}{5}) = t^4 - \frac{6}{7}t^2 + \frac{3}{35}.$$

To conclude, we normalise these vectors, obtaining

$$\frac{1}{\sqrt{2}}, \frac{\sqrt{3}t}{\sqrt{2}}, \frac{\sqrt{5}(3t^2-1)}{2\sqrt{2}}, \frac{\sqrt{7}(5t^3-3t)}{2\sqrt{2}}, \frac{3(35t^4-30t^2+3)}{8\sqrt{2}}$$

4. (**a**) The first two formulas are manifestly bilinear, the third one is not since

$$\mathrm{tr}(A+B_1+B_2) = \mathrm{tr}(A) + \mathrm{tr}(B_1) + \mathrm{tr}(B_2) \neq \mathrm{tr}(A) + \mathrm{tr}(B_1) + \mathrm{tr}(A) + \mathrm{tr}(B_2) = \mathrm{tr}(A+B_1) + \mathrm{tr}(A+B_2),$$

the fourth one is not bilinear since $\det(2AB) = 4\det(AB) \neq 2\det(AB)$.

(**b**) All of them are symmetric: the first one is because of the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ proved in the first semester, the second one because $BA^T = (AB^T)^T$, the third one because $A+B = B+A$, the fourth one because $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

(**c**) For $A = B$, the first one becomes $\mathrm{tr}(A^2)$ which is not always nonnegative, e.g. for $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the second one is the sum of squares of entries of A, so is positive, the third one is just $\mathrm{tr}(2A)$ so clearly is not positive, the fourth one is $\det(A^2) = \det(A)^2$ which is nonnegative but vanishes for many nonzero matrices, e.g. for $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

5. We have

$$|\mathbf{v}+\mathbf{w}|^2 = (\mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}) = (\mathbf{v},\mathbf{v}) + 2(\mathbf{v},\mathbf{w}) + (\mathbf{w},\mathbf{w}),$$

which is less than

$$(\mathbf{v},\mathbf{v}) + 2|\mathbf{v}||\mathbf{w}| + (\mathbf{w},\mathbf{w}) = (|\mathbf{v}| + |\mathbf{w}|)^2$$

by Cauchy–Schwartz inequality, so we get the statement of the problem after extracting square roots.