MA 1112: Linear Algebra II
Homework problems for March 19, 2019

Solutions to this problem sheet are to be handed in after our class at 5pm on Tuesday March 19 (because Monday is a holiday).
Please attach a cover sheet with a declaration `http://tcd-ie.libguides.com/plagiarism/declaration` confirming that you know and understand College rules on plagiarism. On the same cover sheet, please put your name, student number, and name of the degree (Maths/TP/TSM), and staple all the sheets together. (Failure to do that may result in misplaced/lost sheets, for which no responsibility can be taken by instructors.)

1. Which of the following formulas define scalar products on $\mathbb{R}^2$? Explain your answer.

(**a**) $(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}) = x_1y_2+x_2y_1$;

(**b**) $(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}) = x_1x_2$;

(**c**) $(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}) = x_1x_2+7y_1y_2$;

(**d**) $(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}) = x_1x_2+x_1y_2+x_2y_1+y_1y_2$;

(**e**) $(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}) = x_1x_2+x_1y_2+y_1y_2$;

2. For the space $\mathbb{R}^3$ with the standard inner product, find the orthogonal basis e_1, e_2, e_3 obtained by Gram–Schmidt orthogonalisation from $f_1=\begin{pmatrix}2\\1\\0\end{pmatrix}$, $f_2=\begin{pmatrix}0\\2\\1\end{pmatrix}$, $f_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

3. For the vector space of all polynomials in t of degree at most 4 and the scalar product on this space given by

$$(p(t),q(t))=\int_{-1}^{1}p(t)q(t)\,dt,$$

find the result of Gram–Schmidt orthogonalisation of the vectors $1, t, t^2, t^3, t^4$.

4. For the vector space V of all 2×2-matrices with real entries, consider the following four functions with two arguments:

$$(A,B)_1=\mathrm{tr}(AB),\quad (A,B)_2=\mathrm{tr}(AB^T),\quad (A,B)_3=\mathrm{tr}(A+B),\quad (A,B)_4=\det(AB).$$

Which of them are (**a**) bilinear? (**b**) symmetric? (**c**) positive (nonnegative for $A=B$ and only zero for $A=B$ when $A=B=0$)? Show that only the second one defines a scalar product.

5. Show that for two vectors $\mathbf{v}$ and $\mathbf{w}$ of a Euclidean vector space V we have

$$|\mathbf{v}+\mathbf{w}|\leqslant|\mathbf{v}|+|\mathbf{w}|.$$

(*Hint:* use definition of length and Cauchy–Schwartz.)