MA1112: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 13

Euclidean spaces

Informally, a Euclidean space is a vector space with a scalar product. Let us formulate a precise definition. In this lecture, we shall assume that our scalars are real numbers.

Definition 1. A vector space V is said to be a Euclidean space if it is equipped with a function (scalar product) $V \times V \to \mathbb{R}$, $v_1, v_2 \mapsto (v_1, v_2)$ satisfying the following conditions:

- bilinearity: $(c_1 v_1 + c_2 v_2, v) = c_1(v_1, v) + c_2(v_2, v)$ and $(v, c_1 v_1 + c_2 v_2) = c_1(v, v_1) + c_2(v, v_2)$,
- symmetry: $(v_1, v_2) = (v_2, v_1)$ for all v_1, v_2,
- positivity: $(v, v) \geqslant 0$ for all v, and $(v, v) = 0$ only for $v = 0$.

Example 1. Let $V = \mathbb{R}^n$ with the *standard scalar product*

$$(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

All the three properties are trivially true.

Example 2. Let V be the vector space of continuous functions on $[0, 1]$, and

$$(f(t), g(t)) = \int_0^1 f(t) g(t)\, dt.$$

The symmetry is obvious, the bilinearity follows from linearity of the integral, and the positivity follows from the fact that if $\int_0^1 h(t)\, dt = 0$ for a nonnegative continuous function $h(t)$, then $h(t) = 0$.

Lemma 1. *For every scalar product and every basis $e_1, \ldots, e_n$ of V, we have*

$$(x_1 e_1 + \ldots + x_n e_n, y_1 e_1 + \ldots + y_n e_n) = \sum_{i,j=1}^{n} a_{ij} x_i y_j,$$

where $a_{ij} = (e_i, e_j)$.

This follows immediately from the bilinearity property of scalar products.

Orthonormal bases

A system of vectors $e_1, \ldots, e_k$ of a Euclidean space V is said to be orthogonal, if it consists of nonzero vectors, which are pairwise orthogonal: $(e_i, e_j) = 0$ for $i \neq j$. An orthogonal system is said to be orthonormal, if all its vectors are of length 1: $(e_i, e_i) = 1$. Note that a basis $e_1, \ldots, e_n$ of V is orthonormal if and only if

$$(x_1 e_1 + \ldots + x_n e_n, y_1 e_1 + \ldots + y_n e_n) = x_1 y_1 + \ldots + x_n y_n$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n$.

In other words, an orthonormal basis provides us with a system of coordinates that identifies V with $\mathbb{R}^n$ with the standard scalar product.

Lemma 2. *An orthonormal system is linearly independent.*

Proof. Indeed, assuming $c_1e_1+\ldots+c_ke_k=0$, we have

$$0=(0,e_p)=(c_1e_1+\ldots+c_ke_k,e_p)=c_1(e_1,e_p)+\ldots+c_k(e_k,e_p)=c_p(e_p,e_p)=c_p.$$

□

Theorem 1. *Every finite-dimensional Euclidean space has an orthonormal basis.*

Proof. We shall describe a process called *Gram–Schmidt orthogonalisation procedure* which starts from some basis $f_1,\ldots,f_n$, and transform it into an orthogonal basis which we then make orthonormal. Namely, we shall prove by induction that there exists a basis $e_1,\ldots,e_{k-1},f_k,\ldots,f_n$, where the first $(k-1)$ vectors form an orthogonal system and are equal to linear combinations of the first $(k-1)$ vectors of the original basis. For $k=1$ the statement is empty, so there is nothing to prove. Assume that our statement is proved for some k, and let us show how to deduce it for $k+1$. Let us search for e_k of the form $f_k-a_1e_1-\ldots-a_{k-1}e_{k-1}$; this way the condition on linear combinations on the first k vectors of the original basis is automatically satisfied. Conditions $(e_k,e_j)=0$ for $j=1,\ldots,k-1$ mean that

$$0=(f_k-a_1e_1-\ldots-a_{k-1}e_{k-1},e_j)=(f_k,e_j)-a_1(e_1,e_j)-\ldots-a_{k-1}(e_{k-1},e_j),$$

and the induction hypothesis guarantees that the latter is equal to

$$(f_k,e_j)-a_j(e_j,e_j),$$

so we can put $a_j=\frac{(f_k,e_j)}{(e_j,e_j)}$ for all $j=1,\ldots,k-1$. Clearly, the linear span of the vectors $e_1,\ldots,e_{k-1},f_k,\ldots,f_n$ is the same as the linear span of the vectors $e_1,\ldots,e_{k-1},e_k,f_{k+1},\ldots,f_n$ (because we can recover the original set back: $f_k=e_k+a_1e_1+\ldots+a_{k-1}e_{k-1}$). Therefore, $e_1,\ldots,e_{k-1},e_k,f_{k+1},\ldots,f_n$ are n vectors in an n-dimensional vector space that form a spanning set; they also must form a basis.

To complete the proof, we normalise all vectors, replacing each e_k by $\frac{1}{\sqrt{(e_k,e_k)}}e_k$. □

This theorem effectively says that every finite-dimensional Euclidean space can be, with a "wise" choice of a coordinate system, identified with the vector space $\mathbb{R}^n$ equipped with its standard scalar product. The key point of linear algebra is exactly that: an informed choice of a coordinate system can simplify things quite significantly.