MA1112: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 15

Orthogonal complements

Lemma 1. *For every subspace U, we have $U \cap U^\perp = \{0\}$.*

Proof. Indeed, if $u \in U \cap U^\perp$, we have $(u,u) = 0$, so $u = 0$. □

Lemma 2. *For every finite-dimensional subspace $U \subset V$, we have $V = U \oplus U^\perp$. (This justifies the name "orthogonal complement" for $U^\perp$.)*

Proof. Let $e_1, \ldots, e_k$ be an orthonormal basis of U. To prove that the direct sum coincides with V, it is enough to prove $V = U + U^\perp$, or in other words that every vector $v \in V$ can be represented in the form $u + u^\perp$, where $u \in U$, $u^\perp \in U^\perp$. Equivalently, we need to represent v in the form $c_1e_1 + \ldots + c_ke_k + u^\perp$, where $c_1, \ldots, c_k$ are unknown coefficients. Computing scalar products with e_j for $j = 1, \ldots, k$, we get a system of equations to determine c_i:

$$(c_1e_1 + \ldots + c_ke_k + u^\perp, e_j) = (v, e_j).$$

Due to orthonormality of our basis and the definition of the orthogonal complement, the left hand side of this equation is c_j. On the other hand, it is easy to see that for every v, the vector

$$v - (v, e_1)e_1 - \ldots, (v, e_k)e_k$$

is orthogonal to all e_j, and so to all vectors from U, and so belongs to $U^\perp$. □

Corollary 1 (Bessel's inequality)**.** *For any vector $v \in V$ and any orthonormal system $e_1, \ldots, e_k$ (not necessarily a basis) we have*

$$(v, v) \geqslant (v, e_1)^2 + \ldots + (v, e_k)^2.$$

Proof. Indeed, we can take $U = \text{span}(e_1, \ldots, e_k)$ and represent $v = u + u^\perp$. Then

$$(v, v) = (u + u^\perp, u + u^\perp) = (u, u) + (u^\perp, u^\perp)$$

because $(u, u^\perp) = 0$, so

$$|v|^2 = |u|^2 + |u^\perp|^2 \geqslant |u|^2 = (u, e_1)^2 + \ldots + (u, e_k)^2 = (v, e_1)^2 + \ldots + (v, e_k)^2.$$

□

An application of Bessel's inequality

Let us consider the Euclidean space of all continuous functions on $[-1, 1]$ with the scalar product

$$(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)\,dt.$$

Let us check that the functions

$$e_1 = \sin \pi t, \ldots, e_n = \sin \pi n t$$

form an orthonormal system there. We have

$$(e_k, e_l) = \int_{-1}^{1} \sin(k\pi t)\sin(l\pi t)\,dt = \int_{-1}^{1} \frac{1}{2}(\cos((k-l)\pi t) - \cos((k+l)\pi t))\,dt = \begin{cases} 0, k \neq l, \\ 1, k = l, \end{cases}$$

because $\int_{-1}^{1} \cos(m\pi t)\,dt = \frac{\sin(m\pi t)}{m}\Big|_{-1}^{1} = 0$ for $m \neq 0$.

Let us now consider the function $h(t) = t$. We have

$$(h(t), h(t)) = \frac{2}{3},$$
$$(h(t), e_k) = \frac{2(-1)^{k+1}}{k\pi},$$

(the latter integral requires integration by parts to compute it), so Bessel's inequality implies that

$$\frac{2}{3} \geqslant \frac{4}{\pi^2} + \frac{4}{4\pi^2} + \frac{4}{9\pi^2} + \ldots + \frac{4}{n^2\pi^2},$$

which can be rewritten as

$$\frac{\pi^2}{6} \geqslant 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2}.$$

Actually $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, which was first proved by Euler. We are not able to establish it here, but it is worth mentioning that Bessel's inequality gives a sharp bound for this sum.