MA1112: Linear Algebra II

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Lecture 18

Suppose we are given a basis $e_1, \ldots, e_n$ of a vector space V, and that b is a symmetric bilinear form on V. Let us denote by B the matrix whose entries are $b(e_i, e_j)$, and by B_k the top left corner submatrix of B. We put $\Delta_k := \det(B_k)$ for $1 \leqslant k \leqslant n$.

Theorem 1 (Jacobi theorem)**.** *Suppose that for all $i = 1, \ldots, n$ we have $\Delta_i \neq 0$. Then there exists a basis $f_1, \ldots, f_n$ where*

$$q(x_1 f_1 + \cdots + x_n f_n) = \frac{1}{\Delta_1} x_1^2 + \frac{\Delta_1}{\Delta_2} x_2^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} x_n^2.$$

Proof. We shall look for a basis of the form

$$\begin{aligned} f_1 &= \alpha_{11} e_1, \\ f_2 &= \alpha_{12} v e_1 + \alpha_{22} e_2, \\ &\ldots, \\ f_n &= \alpha_{1n} e_1 + \alpha_{2n} e_2 + \cdots + \alpha_{nn} e_n. \end{aligned}$$

If we write the conditions $b(f_i, f_j) = 0$ for $i \neq j$ directly, we shall obtain a system of quadratic equations in the unknowns α_{ij}, which is difficult to solve directly. For that reason, we shall use a clever shortcut.

Suppose that we found a basis of the form given above, for which

$$b(f_i, e_j) = 0 \text{ for } j = 1, \ldots, i-1.$$

We shall now verify that these conditions imply $b(f_i, f_j) = 0$ for $i \neq j$. Indeed, for $i > j$ we have

$$b(f_i, f_j) = b(f_i, \alpha_{1j} e_1 + \alpha_{2j} e_2 + \ldots + \alpha_{jj} e_j) = \alpha_{1j} b(f_i, e_1) + \cdots + \alpha_{jj} b(f_i, e_j) = 0,$$

and for $i < j$ we have $b(f_i, f_j) = b(f_j, f_i) = 0$.

For a given i, the conditions

$$b(f_i, e_j) = 0 \text{ for } j = 1, \ldots, i-1$$

form a system of linear equations with i unknowns and $i-1$ equations, so there will inevitably be free unknowns. To normalise the solution, let us also include the equation

$$b(f_i, e_i) = 1.$$

Then the corresponding system of equation becomes

$$\begin{cases} b(e_1, e_1)\alpha_{1i} + b(e_2, e_1)\alpha_{2i} + \ldots + b(e_i, e_1)\alpha_{ii} = 0, \\ b(e_1, e_2)\alpha_{1i} + b(e_2, e_2)\alpha_{2i} + \ldots + b(e_i, e_2)\alpha_{ii} = 0, \\ \ldots \\ b(e_1, e_{i-1})\alpha_{1i} + b(e_2, e_{i-1})\alpha_{2i} + \ldots + b(e_i, e_{i-1})\alpha_{ii} = 0, \\ b(e_1, e_i)\alpha_{1i} + b(e_2, e_i)\alpha_{2i} + \ldots + b(e_i, e_i)\alpha_{ii} = 1. \end{cases}$$

The matrix of the this system of equation is $B_i^T = B_i$, so by our assumption this system has just one solution for each $i = 1, \dots, n$. This already ensures that under the constraints we imposed the basis $f_1, \dots, f_n$ is unique, and the matrix of the bilinear form b relative to this basis is diagonal. Let us compute the diagonal entries $b(f_i, f_i)$. We have

$$b(f_i, f_j) = b(f_i, \alpha_{1j}e_1 + \alpha_{2j}e_2 + \dots + \alpha_{ii}e_i) = \alpha_{1j}b(f_i, e_1) + \dots + \alpha_{ii}b(f_i, e_i) = \alpha_{ii}.$$

To compute α_{ii}, we use the Cramer's rule for solving systems of linear equations. The last unknown is equal to the ratio $\frac{\det(B_{ii})}{\det(B_i)}$, where B_{ii} is obtained by B_i by replacing the last column by the right hand side of the given system of equations. Expanding that determinant along its rightmost column, we get $\alpha_{ii} = \frac{\Delta_{i-1}}{\Delta_i}$ for $i > 1$, and $\alpha_{11} = \frac{1}{\Delta_1}$, as required. □

Note that the Jacobi theorem has this requirement that all Δ_i are nonzero, which heavily depends on the choice of basis. Thus, it is not always applicable. However, there are some important instances where it is useful, including the proof of the next result.

Sylvester's criterion

Theorem 2 (Sylvester's criterion)**.** *The given symmetric bilinear form is positive definite if and only if*

$$\Delta_k > 0 \quad \textit{for all} \quad k = 1, \dots, n.$$

Proof. Suppose that all Δ_k are positive. Then in particular they are all non-zero, and we are in the situation of Jacobi theorem, which immediately shows that b is positive definite, since $q(v) = b(v, v)$ is represented by a sum of squares of coordinates with positive coefficients.

Suppose that b is positive definite. Let us show that it is impossible to have $\Delta_k = 0$ for some k. Assume the contrary. Then the homogeneous system of linear equations

$$\begin{cases} b(e_1, e_1)x_1 + b(e_2, e_1)x_2 + \dots + b(e_k, e_1)x_k = 0, \\ b(e_1, e_2)x_1 + b(e_2, e_2)x_2 + \dots + b(e_k, e_2)x_k = 0, \\ \dots \\ b(e_1, e_k)x_1 + b(e_2, e_k)x_2 + \dots + b(e_k, e_k)x_k = 0 \end{cases}$$

has a nontrivial solution. Let us take this solution $c_1, \dots, c_k$, and consider the vector $v = c_1e_1 + \dots + c_ke_k$. We have

$$b(v, e_i) = b(e_1, e_i)c_1 + b(e_2, e_i)c_2 + \dots + b(e_k, e_i)c_k = 0 \text{ for } i = 1, \dots, k.$$

Therefore,

$$b(v, v) = c_1b(v, e_1) + c_2b(v, e_2) + \dots + c_kb(v, e_k) = 0,$$

which, coupled with positivity of b, implies $c_1 = \dots = c_k = 0$, which is a contradiction. Therefore, all the determinants Δ_k are nonzero, and then the previous theorem implies that they must be positive, or else the expansion

$$q(x_1f_1 + \dots + x_nf_n) = \frac{1}{\Delta_1}x_1^2 + \frac{\Delta_1}{\Delta_2}x_2^2 + \dots + \frac{\Delta_{n-1}}{\Delta_n}x_n^2$$

has a negative coefficient, and so b cannot be positive definite. □

A bilinear form b is said to be *negative definite* if $b(v, v) < 0$ for every nonzero vector v. From the Sylvester criterion, we immediately deduce that b is negative definite if $(-1)^k\Delta_k$ is positive for all k. Indeed, we look at the form $-b$ which must be positive; for this form the corresponding determinant is $(-1)^k\Delta_k$.

Our last step would be to prove a theorem relating linear transformations and bilinear forms. Suppose that B is a real symmetric matrix representing a bilinear form b.

Theorem 3. *All the eigenvalues of B are real numbers. Moreover, the signature of b is completely determined by eigenvalues of B: the number n_+ is the number of positive eigenvalues, the number n_- is the number of negative eigenvalues, and the number n_0 is the number of zero eigenvalues.*