MA1112: Linear Algebra II

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Lecture 1

Administrative notes

We shall have three hours of classes per week, at 10am and 11am on Mondays and at 5pm on Tuesdays. Most of those classes will be lectures, and some of the Tuesday classes will be tutorials (more dense around "practical" topics, and less dense around "theoretical topics"). There will be weekly home assignments, due by the end of the 11am class on Mondays. Late assignments will not be accepted.

The mark for this module will be made of 60% of the final exam mark, 20% of the average of homework marks, and 20% of the midterm test. The midterm test will take place on Monday February 25, just **before** the reading week, it will last for about 1.5 hours (arrangements to be confirmed).

Recollections from the module 1111

In this module, we shall be using extensively notions and methods from the module 1111. Please consult the notes for that module when you have questions. The most important notions that you should be fairly fluent in are that of

- vector space, linearly independent set of vectors, spanning set of vectors, basis
- linear map and linear transformation, matrices representing them
- transition matrix, change of coordinates under change of basis, change of matrix of a linear map under change of basis

Outline of this module

In the end of module 1111, we discussed making a matrix of a linear transformation diagonal, using a basis of eigenvectors (when exists). The main question that we shall address this semester is

> Given a linear map or a linear transformation, how "simple" can its matrix be made by a change of coordinates?

We shall find a number of answers to this question, and see some applications.

The module roughly consists of two parts:

- Jordan decomposition theorem (answers the question in full generality — what to do when there is no basis of eigenvectors).
- Case of vector spaces with extra structure (scalar product) and constrained choice of coordinates (orthonormal bases).

It is normal to ask oneself some questions about this outline. First, it might seem a bit decadent to spend a whole module answering just one question! Well, truth is that it is a very major question. A lot of seemingly unrelated questions in mathematics and its application (to computer science, theoretical physics, economics, biology etc.) is about normal forms of matrices of linear maps and linear transformations. Identifying some questions and using standard theorems from linear algebra offers one an extremely powerful arsenal of tools.

Second, it is not immediately clear why one bothers with the second half of the module if the Jordan decomposition theorem already answers the question in full generality. Well, it so happens that the presence of extra structure offers extra insight! A lot of applications of linear algebra happen in the presence of scalar products, and we shall see that it will allow us to use intuition coming from low-dimensional cases in a powerful way.

Two interesting instances of problems we shall be able to solve are as follows.

- image compression (cameras in our smartphones can take pictures of very high quality; however, the price of that is those images take an awful lot of space of the phone's memory... but if we send an image in an instant messenger to our friends, they receive an image that looks more or less the same on the screen, but takes a small fraction of the space! How is that accomplished? we shall talk about a solution to that offered by linear algebra)
- unexpected equalities and inequalities: we shall use scalar products to prove that
$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}\leqslant\frac{\pi^2}{6},$$
a beautiful inequality discovered by Euler (who in fact proved that the infinite sum of inverse squares is equal to $\frac{\pi^2}{6}$).

Kernels and images

Definition 1. Let $\varphi\colon V\to W$ be a linear map between two vector spaces. We define its *kernel* $\operatorname{Ker}(\varphi)$ as the set of all vectors $v\in V$ for which $\varphi(v)=0$, and its image $\operatorname{Im}(\varphi)$ as the set of all vectors $w\in W$ such that $w=\varphi(v)$ for some $v\in V$.

Lemma 1. *The subset* $\operatorname{Ker}(\varphi)$ *is a subset of* V*, and the subset* $\operatorname{Im}(\varphi)$ *is a subspace of* W*.*

Proof. Note that $\varphi(0)=0$ which shows that $0\in\operatorname{Ker}(\varphi)$ and $0\in\operatorname{Im}(\varphi)$, so those subsets are non-empty. (Warning: those two zeros are zeros in two different vector spaces!)

Furthermore,

if $v_1,v_2\in\operatorname{Ker}(\varphi)$, then $\varphi(v_1+v_2)=\varphi(v_1)+\varphi(v_2)=0+0=0$,

if $v\in\operatorname{Ker}(\varphi)$, then for any scalar c we have $\varphi(c\cdot v)=c\varphi(v)=c\cdot 0=0$,

if $w_1,w_2\in\operatorname{Im}(\varphi)$, then $w_1=\varphi(v_1)$ for some $v_1\in V$ and $w_2=\varphi(v_2)$ for some $v_2\in V$, so we have $w_1+w_2=\varphi(v_1)+\varphi(v_2)=\varphi(v_1+v_2)$,

if $w\in\operatorname{Im}(\varphi)$, then $w=\varphi(v)$ for some $v\in V$, so for any scalar c we have $c\cdot w=c\varphi(v)=\varphi(c\cdot v)$. □

Rank and nullity of a linear map

Definition 2. The rank of a linear map φ, denoted by $\operatorname{rk}(\varphi)$, is the dimension of the image of φ. The nullity of φ, denoted by $\operatorname{null}(\varphi)$, is the dimension of the kernel of φ.

Example 1. Let $I\colon V\to V$ be the identity map, so that $I(v)=v$ for each $v\in V$. Then $\operatorname{Ker}(I)=\{0\}$, and $\operatorname{Im}(I)=V$, so that $\operatorname{null}(I)=0$ and $\operatorname{rk}(I)=\dim(V)$.

Example 2. Let $0\colon V\to W$ be the map sending every vector v to $0\in W$: $0(v)=0$ for each $v\in V$. Then $\operatorname{Ker}(0)=V$, and $\operatorname{Im}(0)=\{0\}$, so that $\operatorname{null}(0)=\dim(V)$ and $\operatorname{rk}(0)=0$.

Example 3. Let $\varphi\colon\mathbb{R}^2\to\mathbb{R}^2$ be linear transformation given by $\varphi\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}y\\0\end{pmatrix}$, or in other words, $\varphi(v)=Av$, where $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. The latter formula shows that φ is a linear transformation.

Note that $\operatorname{Ker}(\varphi)$ consists of all vectors $\begin{pmatrix}x\\y\end{pmatrix}$ with $y=0$, so $\operatorname{Ker}(\varphi)=\left\{\begin{pmatrix}x\\0\end{pmatrix}\right\}$, and $\operatorname{null}(\varphi)=1$. Interestingly, by direct inspection we have $\operatorname{Im}(\varphi)=\left\{\begin{pmatrix}y\\0\end{pmatrix}\right\}$, so in this case $\operatorname{Ker}(\varphi)=\operatorname{Im}(\varphi)$, and $\operatorname{rk}(\varphi)=\operatorname{null}(\varphi)=1$.