MA1112: Linear Algebra II

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Lecture 3

Let us mention one consequence of the rank-nullity theorem from the previous class.

Proposition 1. *For any linear map* $\varphi\colon V \to W$*, we have* $\mathrm{rk}(\varphi) \leqslant \min(\dim(V), \dim(W))$.

Proof. We have $\mathrm{rk}(\varphi) \leqslant \dim(W)$ because $\mathrm{Im}(\varphi) \subset W$, and the dimension of a subspace cannot exceed the dimension of the whole space. Also, $\mathrm{rk}(\varphi) = \dim(V) - \mathrm{null}(\varphi) \leqslant \dim(V)$.

(Alternatively, one can argue that being the number of pivots in the reduced row echelon form, the rank cannot exceed either the number of rows or the number of columns, but this proof shows some other useful techniques, so we mention it here). □

As we mentioned last time, the rank-nullity theorem and its proofs actually tells us precisely how to simplify matrices of most general linear maps $\varphi\colon V \to W$. If we allowed to change bases of V and W independently, then the rank is the only invariant: every $m \times n$-matrix A can be brought to the form $\begin{pmatrix} I_l & 0_{(n-l)\times l} \\ 0_{k\times(m-l)} & 0_{(n-l)\times(m-l)} \end{pmatrix}$, where $l = \mathrm{rk}(\varphi)$. However, if we restrict ourselves to linear transformations $\varphi\colon V \to V$, then we can only change one basis, and under the changes we replace matrices A by $C^{-1}AC$, where C is the transition matrix. We know several things that remain the same under this change, e.g. the trace and the determinant, so the story gets much more subtle. We shall discuss it in the following lectures. Today we shall focus on one new structural notion, that of the sum and the direct sum of subspaces of a vector space.

Sums and direct sums

Let V be a vector space. Recall that the *span* of a set of vectors $v_1, \ldots, v_k \in V$ is the set of all linear combinations $c_1v_1 + \ldots + c_kv_k$. It is denoted by $\mathrm{span}(v_1, \ldots, v_k)$. Also, vectors $v_1, \ldots, v_k$ are linearly independent if and only if they form a basis of their linear span. Our next definition provides a generalisation of these two sentences to the case of arbitrary subspaces, rather than vectors.

Definition 1. Let $V_1, \ldots, V_k$ be subspaces of V. Their *sum* $V_1 + \ldots + V_k$ is defined as the set of vectors of the form $v_1 + \ldots + v_k$, where $v_1 \in V_1, \ldots, v_k \in V_k$. The sum of the subspaces $V_1, \ldots, V_k$ is said to be *direct* if $0 + \ldots + 0$ is the only way to represent $0 \in V_1 + \ldots + V_k$ as a sum $v_1 + \ldots + v_k$. In this case, it is denoted by $V_1 \oplus \ldots \oplus V_k$.

Lemma 1. $V_1 + \ldots + V_k$ *is a subspace of* V.

Proof. It is sufficient to check that $V_1 + \ldots + V_k$ is closed under addition and multiplication by numbers. Clearly,

$$(v_1 + \ldots + v_k) + (v_1' + \ldots + v_k') = ((v_1 + v_1') + \ldots + (v_k + v_k'))$$

and

$$c(v_1 + \ldots + v_k) = ((cv_1) + \ldots + (cv_k)),$$

and the lemma follows, since each V_i is a subspace and hence closed under the vector space operations. □

Example 1. Consider the subspaces U_n and U_m of $\mathbb{R}^{n+m}$, the first one being the linear span of the first n standard unit vectors, and the second one being the linear span of the last m standard unit vectors. We have $\mathbb{R}^{n+m} = U_n + U_m = U_n \oplus U_m$.

Example 2. For a collection of nonzero vectors $v_1, \dots, v_k \in V$, consider the subspaces $V_1, \dots, V_k$, where V_i consists of all multiples of v_i. Then, clearly, $V_1 + \dots + V_k = \text{span}(v_1, \dots, v_k)$, and this sum is direct if and only if the vectors v_j are linearly independent.

Example 3. For two subspaces V_1 and V_2, their sum is direct if and only if $V_1 \cap V_2 = \{0\}$. Indeed, if $v_1 + v_2 = 0$ is a nontrivial representation of 0, $v_1 = -v_2$ is in the intersection, and vice versa. The situation of three and more subspaces is more subtle.

Theorem 1. *If V_1 and V_2 are subspaces of V, we have*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

In particular, the sum of V_1 and V_2 is direct if and only if $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$.

It is important to note that an analogue of this theorem for three or more subspaces is not available, contrary to what intuition from combinatorics of finite sets may suggest to us.

Proof. Let us pick a basis $e_1, \dots, e_p$ of the intersection $V_1 \cap V_2$, and extend this basis to a bigger set of vectors in two different ways, one way obtaining a basis of V_1, and the other way — a basis of V_2. Let $e_1, \dots, e_k, f_1, \dots, f_q$ and $e_1, \dots, e_k, g_1, \dots, g_r$ be the resulting bases of V_1 and V_2 respectively. Let us prove that

$$e_1, \dots, e_p, f_1, \dots, f_q, g_1, \dots, g_r$$

is a basis of $V_1 + V_2$. It is a complete system of vectors, since every vector in $V_1 + V_2$ is a sum of a vector from V_1 and a vector from V_2, and vectors there can be represented as linear combinations of $e_1, \dots, e_p, f_1, \dots, f_q$ and $e_1, \dots, e_p, g_1, \dots, g_r$ respectively. To prove linear independence, let us assume that

$$a_1 e_1 + \dots + a_p e_p + b_1 f_1 + \dots + b_q f_q + c_1 g_1 + \dots + c_r g_r = 0.$$

Rewriting this formula as $a_1 e_1 + \dots + a_p e_p + b_1 f_1 + \dots + b_q f_q = -(c_1 g_1 + \dots + c_r g_r)$, we notice that on the left we have a vector from V_1 and on the right a vector from V_2, so both the left hand side and the right hand side is a vector from $V_1 \cap V_2$, and so can be represented as a linear combination of $e_1, \dots, e_p$ alone. However, the vectors on the right hand side together with e_i form a basis of V_2, so there is no nontrivial linear combination of these vectors that is equal to a linear combination of e_i. Consequently, all coefficients c_i are equal to zero, so the left hand side is zero. This forces all coefficients a_i and b_i to be equal to zero, since $e_1, \dots, e_p, f_1, \dots, f_q$ is a basis of V_1. This completes the proof of the linear independence of the vectors $e_1, \dots, e_p, f_1, \dots, f_q, g_1, \dots, g_r$.

Summing up, $\dim(V_1) = p + q$, $\dim(V_2) = p + r$, $\dim(V_1 + V_2) = p + q + r$, $\dim(V_1 \cap V_2) = p$, and our theorem follows. $\square$

In practice, it is important sometimes to determine the intersection of two subspaces, each presented as a linear span of several vectors. This question naturally splits into two different questions.

First, it makes sense to find a basis of each of these subspaces. To determine a basis for a linear span of given vectors, the easiest way is to form the matrix whose columns are the given vectors, and find its reduced column echelon form (like the reduced row echelon form, but with elementary operations on columns). Nonzero columns of the result form a basis of the linear span subspace.

Once we know a basis $v_1, \dots, v_k$ for the first subspace, and a basis $w_1, \dots, w_l$ for the second one, the question reduces to solving the linear system $c_1 v_1 + \dots + c_k v_k = d_1 w_1 + \dots + d_l w_l$. For each solution to this system, the vector $c_1 v_1 + \dots + c_k v_k$ is in the intersection, and vice versa. Computationally, this agrees well with the first step, because computing the reduced column echelon form produces a system of equations with many zero entries already. We shall discuss an example of a computation like that in the next lecture.