MA1112: Linear Algebra II

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Lecture 4

In practice, it is important sometimes to determine the intersection of two subspaces, each presented as a linear span of several vectors. The usual set-up for this question is as follows. We take a vector space V, and its two subspaces U_1 and U_2, where $U_1 = \mathrm{span}(e_1, \ldots, e_k)$ and $U_2 = \mathrm{span}(f_1, \ldots, f_l)$. The goal is to describe a basis of $U_1 \cap U_2$.

First, it makes sense to find a "convenient" basis of each of these subspaces. For that, we choose a basis of the ambient space V, and associate to each subspace U_i the matrix M_i whose columns are columns of coordinates of the spanning set with respect to that basis. (In case of $V = \mathbb{R}^n$, this task is trivial, since everything is already written with respect to the basis of standard unit vectors.) Then for each of these two matrices, we compute its reduced column echelon form (like the reduced row echelon form, but with elementary operations on columns). Nonzero columns of the result form a basis of the linear span.

Once we know a basis $g_1, \ldots, g_p$ for the first subspace, and a basis $h_1, \ldots, w_q$ for the second one, the question reduces to solving the linear system $a_1 g_1 + \ldots + a_p g_p = b_1 h_1 + \ldots + b_q h_q$. For each solution to this system, the vector $a_1 g_1 + \ldots + a_p g_p$ is in the intersection, and vice versa. Computationally, the fact that we computed the reduced column echelon forms before, we have a system with many zero coefficients which is quite easy to solve.

Example 1. Let us consider two following subspaces of $V = \mathbb{R}^5$: the subspace U_1 is the span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 5 \\ -1 \\ -1 \\ 14 \end{pmatrix}$, and subspace U_2 is the span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ -2 \\ -3 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.

Let us first find the bases of these subspaces. As we mentioned, we should compute reduced column echelon forms, but to not introduce new notation, we shall use transpose matrices M_1^t and M_2^t and row echelon forms. The computations are as follows:

$$M_1^t = \begin{pmatrix} 2 & 1 & 0 & -4 & 2 \\ -4 & 1 & 3 & -1 & 2 \\ 0 & 5 & -1 & -1 & 14 \end{pmatrix} \overset{\frac12(1),(2)+4(1)}{\longmapsto} \begin{pmatrix} 1 & \frac12 & 0 & -2 & 1 \\ 0 & 3 & 3 & -9 & 6 \\ 0 & 5 & -1 & -1 & 14 \end{pmatrix} \overset{\frac13(2),(3)-5(2)}{\longmapsto}$$

$$\begin{pmatrix} 1 & \frac12 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & -16 & 44 & 16 \end{pmatrix} \overset{1-\frac12(2),-\frac{1}{16}(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & -\frac12 & -\frac12 & 0 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & 1 & -\frac{11}{4} & 1 \end{pmatrix} \overset{(2)-(3),1+\frac12(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & -\frac{15}{8} & \frac12 \\ 0 & 1 & 0 & -\frac14 & 1 \\ 0 & 0 & 1 & -\frac{11}{4} & 1 \end{pmatrix},$$

$$M_2^t = \begin{pmatrix} 2 & 1 & 0 & 1 & 1 \\ 2 & -1 & -2 & -3 & -1 \\ 1 & 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix} \overset{\frac12(1),(2)-2(1),(3)-(1)}{\longmapsto} \begin{pmatrix} 1 & \frac12 & 0 & \frac12 & \frac12 \\ 0 & -2 & -2 & -4 & -2 \\ 0 & -\frac12 & -2 & -\frac52 & \frac32 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix} \overset{-\frac12(2),(1)-\frac12(2),(3)+\frac12(2),(4)-(2)}{\longmapsto}$$

$$\begin{pmatrix} 1 & 0 & -\frac12 & -\frac12 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -\frac32 & -\frac32 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{-\frac23(3),(1)+\frac12(3),(2)-(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac23 \\ 0 & 1 & 0 & 1 & \frac73 \\ 0 & 0 & 1 & 1 & -\frac43 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This means that each of the subspaces is three-dimensional. To compute the intersection, recall that by definition the intersection consists of all vectors that belong to both of the subspaces. Let us denote by g_1, g_2, g_3 the basis vectors for U_1 we just found, and by h_1, h_2, h_3 the basis vectors for U_2. Then the intersection consists of all vectors v that can be represented in the form

$$v = a_1 g_1 + a_2 g_2 + a_3 g_3 = b_1 h_1 + b_2 h_2 + b_3 h_3$$

for some $a_1, a_2, a_3, b_1, b_2, b_3$. This is a homogeneous system of linear equations with unknowns $a_1, a_2, a_3, b_1, b_2, b_3$. Its matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -15/8 & -1/4 & -11/4 & 0 & -1 & -1 \\ 1/2 & 1 & 1 & 2/3 & -7/3 & 4/3 \end{pmatrix}$$

Let us bring this matrix to the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -\frac{15}{8} & -\frac{1}{4} & -\frac{11}{4} & 0 & -1 & -1 \\ \frac{1}{2} & 1 & 1 & \frac{2}{3} & -\frac{7}{3} & \frac{4}{3} \end{pmatrix} \overset{(4)+\frac{15}{8}(1),5-\frac{1}{2}(1)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -\frac{1}{4} & -\frac{11}{4} & -\frac{15}{8} & -1 & -1 \\ 0 & 1 & 1 & \frac{7}{6} & -\frac{7}{3} & \frac{4}{3} \end{pmatrix} \overset{(4)+\frac{1}{4}(2),(5)-(2)}{\longmapsto}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -\frac{11}{4} & -\frac{15}{8} & -\frac{5}{4} & -1 \\ 0 & 0 & 1 & \frac{7}{6} & -\frac{4}{3} & \frac{4}{3} \end{pmatrix} \overset{(4)+\frac{11}{4}(3),(5)-(3)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{15}{8} & -\frac{5}{4} & -\frac{15}{4} \\ 0 & 0 & 0 & \frac{7}{6} & -\frac{4}{3} & \frac{7}{3} \end{pmatrix} \overset{-\frac{8}{15}(4),(5)-\frac{7}{6}(4)}{\longmapsto}$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & -\frac{19}{9} & 0 \end{pmatrix} \overset{-\frac{9}{19}(5),(4)-\frac{2}{3}(5),(2)+(5)}{\longmapsto} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We see that the last unknown is free, so the general solution is $a_1 = -2t$, $a_2 = 0$, $a_3 = t$, $b_1 = -2t$, $b_2 = 0$, $b_3 = t$, and the intersection can be described as the set of all vectors of the form

$$t(-2u_1 + u_3) = t(-2w_1 + w_3),$$

so for a basis of the intersection we can take $-2u_1 + u_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

More generally, if we had more than one free variables, this procedure would give us a vector of the form $t_1 v_1 + \cdots + t_m v_m$, where t_i are all the free variables; and $v_1, \ldots, v_m$ form a basis of the intersection.