MA1112: Linear Algebra II

Dr. Vladimir Dotsenko (Vlad)

Lecture 5

Another important calculation which we shall be doing quite a bit in the following classes is computing a basis of a vector space relative to its subspace. (Once again, we assume the spaces presented as linear spans of several vectors.)

The general set-up here is as follows. We have the ambient vector space V, inside it a subspace $W = \text{span}(e_1, \ldots, e_k)$, and then a subspace $W' = \text{span}(f_1, \ldots, f_l)$ of W. In this case, it is reasonable to bring the matrix of vectors spanning W' to its reduced column echelon form, and then reduce the matrix of vectors spanning W with respect to the thus obtained reduced column echelon matrix using the pivots of the latter. The resulting matrix then should be brought to its reduced column echelon form, giving a relative basis.

Example 1. Consider the subspace W of $\mathbb{R}^5$ equal to the space U_2 from the previous class, that is the span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ -2 \\ -3 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$. Let us also define the subspace W' as the span of the vectors $\begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}$.

Let us first find a "convenient" basis of W'. Using transpose matrices again, we perform the row operations

$$\begin{pmatrix} 1 & -1 & 3 & 2 & -7 \\ 3 & 1 & 1 & 2 & -1 \end{pmatrix} \overset{(2)-3(1),1/4(2)}{\mapsto} \begin{pmatrix} 1 & -1 & 3 & 2 & -7 \\ 0 & 1 & -2 & -1 & 5 \end{pmatrix} \overset{(1)+(2)}{\mapsto} \begin{pmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 1 & -2 & -1 & 5 \end{pmatrix}.$$

Recall that the basis of W is given by the transpose of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1 & 7/3 \\ 0 & 0 & 1 & 1 & -4/3 \end{pmatrix}.$$

From this, it is already clear that the rows of the former matrix are $r_1 + r_3$ and $r_2 - 2r_3$, where r_i are the rows of the latter matrix, so W' is indeed a subspace of W. Let us now reduce rows of W with respect to rows of W':

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1 & 7/3 \\ 0 & 0 & 1 & 1 & -4/3 \end{pmatrix} \overset{(1)-(1'),(2)-(2')}{\mapsto} \begin{pmatrix} 0 & 0 & -1 & -1 & 4/3 \\ 0 & 0 & 2 & 2 & -8/3 \\ 0 & 0 & 1 & 1 & -4/3 \end{pmatrix}.$$

Clearly, the reduced row echelon form of this matrix is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & -4/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so the vector

$$v = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -4/3 \end{pmatrix}$$

can be chosen to form the relative basis, that is a set of linearly independent vectors that, together with a basis of W', give us a basis of W.

Invariant subspaces

Our next step is to introduce a yet another definition that will be needed to study arbitrary linear transformations, that of an invariant subspace.

Definition 1. Let V be a vector space, and $\varphi\colon V \to V$ be a linear transformation. A subspace U of V is said to be *invariant* under φ if $\varphi(U) \subset (U)$, that is $\varphi(u) \in U$ for all $u \in U$.

Example 2. All multiples of an eigenvector of φ form a subspace of V that is invariant under φ. Indeed, all multiples of any vector form a subspace, and if it is an eigenvector, then φ maps any vector from this subspace to its multiple.

Let us use this opportunity to fix some notation related to eigenvectors. Recall that eigenvalues of a linear transformation φ of an n-dimensional space V are roots of $\det(A_{\varphi,e} - tI_n)$, where $e_1, \ldots, e_n$ is any basis of V.

Definition 2. The expression $\det(A_{\varphi,e} - tI_n)$ is called the *characteristic polynomial* of the linear transformation φ. It is often denoted $\chi_\varphi(t)$.

By inspection, $\chi_\varphi(t)$ is a polynomial in t of degree n with leading coefficient $(-1)^n$. Note that over complex numbers every polynomial has a root, and so every linear transformation has an eigenvector.

The example of eigenvectors is, in a sense, a very useful motivation for introducing invariant subspaces. Namely, suppose that $U \subset V$ is an invariant subspace of a linear transformation φ. Let $e_1, \ldots, e_k$ be a basis of φ, and $f_1, \ldots, f_l$ a basis of V relative to U, so that $e_1, \ldots, e_k, f_1, \ldots, f_l$ is a basis of V. Then, by direct inspection, the matrix of the linear transformation φ with respect to this basis has the block-triangular form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A is the matrix describing how φ transforms the invariant subspace U. Our hunt for invariant subspaces is ultimately motivated by a wish to reduce a "big" problem of working with an arbitrary linear transformations to similar but "smaller" ones.

From now on, we shall work with complex numbers as scalars for a while, thus ensuring that linear transformations have many invariant subspaces. This is not true over real numbers: some linear transformations (e.g. rotations in 2D) have no nontrivial invariant subspaces at all.