MA1112: Linear Algebra II

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Lecture 7

Case $\varphi^k = 0$

Suppose that φ is a linear transformation of a vector space V for which $\varphi^k = 0$. We shall adapt the argument that we had for $k = 2$ to this general case. We assume that k is actually the smallest power of φ that vanishes, so that $\varphi^{k-1} \neq 0$.

Let us put, for each p, $N_p = \ker(\varphi^p)$. Of course, we have $N_k = N_{k+1} = N_{k+2} = \ldots = V$.

We shall now construct a basis of V of a very particular form. It will be constructed in k steps. First, we find a basis of $V = N_k$ relative to N_{k-1}. Let $e_1, \ldots, e_s$ be vectors of this basis.

The following result is proved in the same way as the one from the previous class:

Lemma 1. *The vectors* $e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s)$ *are linearly independent relative to* N_{k-2}.

Proof. Indeed, assume that $a_1e_1 + \ldots + a_se_s + b_1\varphi(e_1) + \ldots + b_s\varphi(e_s) \in N_{k-2}$.

Since $e_i \in N_k$, we have $\varphi(e_i) \in N_{k-1}$, so

$$a_1e_1 + \ldots + a_se_s \in -b_1\varphi(e_1) - \ldots - b_s\varphi(e_s) + N_{k-2} \subset N_{k-1},$$

which means that $a_1 = \ldots = a_s = 0$. Thus,

$$\varphi(b_1e_1 + \ldots + b_se_s) = b_1\varphi(e_1) + \ldots + b_s\varphi(e_s) \in N_{k-2},$$

so $b_1e_1 + \ldots + b_se_s \in N_{k-1}$, and we deduce that $b_1 = \ldots = b_s = 0$, thus the lemma follows. □

Now we find vectors $f_1, \ldots, f_t$ which form a basis of N_{k-1} relative to $\operatorname{span}(\varphi(e_1), \ldots, \varphi(e_s)) + N_{k-2}$. Absolutely analogously one can prove

Lemma 2. *The vectors* $e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s), \varphi^2(e_1), \ldots, \varphi^2(e_s), f_1, \ldots, f_t, \varphi(f_1), \ldots, \varphi(f_t)$ *are linearly independent relative to* N_{k-3}.

Proof. Let us assume that

$$a_1^{(1)}e_1 + \ldots + a_s^{(1)}e_s + a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) + a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) + \\ b_1^{(1)}f_1 + \ldots + b_t^{(1)}f_t + b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) \in N_{k-3}.$$

We note that

$$a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) \in N_{k-1},$$
$$a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) \in N_{k-2}$$
$$b_1^{(1)}f_1 + \ldots + b_t^{(1)}f_t \in N_{k-1},$$
$$b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) \in N_{k-2},$$

so we have $a_1^{(1)}e_1 + \ldots + a_s^{(1)}e_s \in N_{k-1}$, and hence $a_1^{(1)} = a_s^{(1)} = 0$ since the vectors e_1, ..., e_s form a relative basis. Thus, we have

$$a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) + a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) + \\ b_1^{(1)}f_1 + \ldots + b_t^{(1)}f_t + b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) \in N_{k-3}.$$

Now,

$$a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) \in \text{span}(\varphi(e_1), \ldots, \varphi(e_s)),$$
$$a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) \in N_{k-2}$$
$$b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) \in N_{k-2},$$

so $b_1^{(1)}f_1 + \ldots + b_t^{(1)}f_t \in \text{span}(\varphi(e_1), \ldots, \varphi(e_s)) + N_{k-2}$, and hence $b_1^{(1)} = b_t^{(1)} = 0$ since the vectors f_1, ..., f_t form a relative basis. Our original assumption simplifies to

$$a_1^{(2)}\varphi(e_1) + \ldots + a_s^{(2)}\varphi(e_s) + a_1^{(3)}\varphi^2(e_1) + \ldots + a_s^{(3)}\varphi^2(e_s) + b_1^{(2)}\varphi(f_1) + \ldots + b_t^{(2)}\varphi(f_t) \in N_{k-3},$$

which can be rewritten as

$$\varphi(a_1^{(2)}e_1 + \ldots + a_s^{(2)}e_s + a_1^{(3)}\varphi(e_1) + \ldots + a_s^{(3)}\varphi(e_s) + b_1^{(2)}f_1 + \ldots + b_t^{(2)}f_t) \in N_{k-3},$$

implying

$$a_1^{(2)}e_1 + \ldots + a_s^{(2)}e_s + a_1^{(3)}\varphi(e_1) + \ldots + a_s^{(3)}\varphi(e_s) + b_1^{(2)}f_1 + \ldots + b_t^{(2)}f_t \in N_{k-2},$$

which by the previous lemma and the relative basis property of f_1, ..., f_t shows that all the coefficients are equal to zero. □

Next we find a basis of N_{k-2} relative to $\text{span}(\varphi^2(e_1), \ldots, \varphi^2(e_s), \varphi(f_1), \ldots, \varphi(f_t)) + N_{k-3}$, etc. We continue that extension process until we end up with a basis of V of the following form:

$$e_1, \ldots, e_s, \varphi(e_1), \ldots, \varphi(e_s), \varphi^2(e_1), \ldots, \varphi^{k-1}(e_1), \ldots, \varphi^{k-1}(e_s),$$
$$f_1, \ldots, f_t, \varphi(f_1), \ldots, \varphi^{k-2}(f_1), \ldots, \varphi^{k-2}(f_t),$$
$$\ldots,$$
$$h_1, \ldots, h_p,$$

where the first line contains several "threads" $e_i, \varphi(e_i), \ldots, \varphi^{k-1}(e_i)$ of length k, the second line — several threads of length $k-1$, ..., the last line — several threads of length 1, that is several vectors from N_1.

Let us rearrange the basis vectors so that vectors forming a thread are all next to each other:

$$e_1, \varphi(e_1), \ldots, \varphi^{k-1}(e_1), \ldots, e_s, \varphi(e_s), \ldots, \varphi^{k-1}(e_s),$$
$$f_1, \varphi(f_1), \ldots, \varphi^{k-2}(f_1), \ldots, f_t, \varphi(f_t), \ldots, \varphi^{k-2}(f_t),$$
$$\ldots,$$
$$g_1, \ldots, g_u.$$

Relative to that basis, the linear transformation φ has the matrix made of *Jordan blocks*

$$J_l = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

one block J_l for each thread of length l.