1112: Linear Algebra II
Selected final exam questions from past years

April 2, 2019

1. (a) Find all eigenvalues and eigenvectors of the matrix
$$B = \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix}.$$
   (b) Find the Jordan normal form of the matrix B from the previous question, and a matrix C that transforms B to its Jordan normal form.
   (c) Find a formula for B^n, and use it to find a formula for the n^{th} term of the sequence defined recursively by $a_0 = 2$, $a_1 = 1$, $a_{n+1} = -4a_n - 4a_{n-1}$.
2. In the vector space of all polynomials in t of degree at most 2 with the scalar product
$$(p(t), q(t)) = \int_{-1}^{1} p(t)q(t)\,dt,$$
find the orthogonal basis which is the output of the Gram-Schmidt orthogonalisation applied to the basis $2+3t$, t^2-1, $t-1$.
3. (a) Formulate the Sylvester's criterion for a quadratic form to be positive definite.
   (b) Determine all values of the parameter a for which the quadratic form
$$q(xe_1 + ye_2 + ze_3) = (18+a)x^2 + 3y^2 + az^2 + 10xy - (8+2a)xz - 4yz$$
   is positive definite.
4. Is the subspace U of $\mathbb{R}^4$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ an invariant subspace of the operator A whose matrix relative to the standard basis is
$$\begin{pmatrix} 0 & 3 & -3 & -1 \\ 1 & 3 & -1 & 0 \\ 7 & 12 & 2 & 3 \\ -3 & -6 & 0 & -1 \end{pmatrix}?$$
Explain your answer.
5. (a) Find all eigenvalues and eigenvectors of the matrix
$$B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$
   (b) Find the Jordan normal form of the matrix B, and a matrix C which is the transition matrix of some Jordan basis of B.

(c) Find a formula for B^n, and use it to find a closed formula for the n^{th} terms of the sequences $\{x_m\}$, $\{y_m\}$ defined recursively as follows:

$$x_0 = 1, y_0 = -5,$$
$$x_{k+1} = x_k - y_k, \quad y_{k+1} = x_k + 3y_k.$$

6. (a) Which bases of a Euclidean space V are called orthogonal? orthonormal?

(b) Show that the $f_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, and $f_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ form a basis of $\mathbb{R}^3$.

(c) Find the orthogonal basis of $\mathbb{R}^3$ which is the output of the Gram-Schmidt orthogonalisation applied to the basis from the previous question. (The scalar product on the $\mathbb{R}^3$ is the standard one.)

7. (a) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

(b) Consider the vector space V of all polynomials in t of degree at most 2. The bilinear form ψ_a on V (depending on a [real] parameter a) is defined by the formula

$$\psi_a(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)(t-a)\,dt.$$

Determine all values of a for which ψ_a is positive definite.

8. (a) Determine the Jordan normal form and find some Jordan basis for the matrix

$$A = \begin{pmatrix} 3 & -4 & 6 \\ 1 & -5 & 3 \\ 0 & -4 & 1 \end{pmatrix}$$

(b) Find a closed formula for A^n.

9. (a) Write down the definition of a Euclidean vector space.

(b) The function $f_a \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ (depending on a real parameter a) is defined by the formula

$$\begin{aligned} f_a(x_1e_1 + x_2e_2 + x_3e_3, y_1e_1 + y_2e_2 + y_3e_3) = \\ = 2x_1y_1 + (x_1y_2 + x_2y_1) + (2a-1)x_2y_2 - \\ - a(x_1y_3 + x_3y_1) - (x_2y_3 + x_3y_2) + x_3y_3 \end{aligned}$$

(here e_1, e_2, e_3 is a basis of $\mathbb{R}^3$). Determine all values of a for which f_a is a scalar product.

10. Let V be a vector space. Show that for every two linear operators $A \colon V \to V$ and $B \colon V \to V$ we have

$$\operatorname{rk}(AB) \leqslant \operatorname{rk}(A) \quad \text{and} \quad \operatorname{rk}(AB) \leqslant \operatorname{rk}(B).$$

Show that if B is invertible, then $\operatorname{rk}(BA) = \operatorname{rk}(A)$, and give an example showing that this equality might hold even if B is not invertible.

11. (a) Determine the Jordan normal form and find some Jordan basis for the matrix

$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix}$$

(b) Find a closed formula for A^n.

12. (a) A quadratic form Q on the space $\mathbb{R}^3$ is defined by the formula

$$Q(xe_1 + ye_2 + ze_3) = (20+4a)x^2 + 12(1+a)xz + 6y^2 + 3z^2.$$

Find all values of the parameter a for which this form is positive definite.

13. A square matrix A (of some size $n \times n$) satisfies the condition $A^2 - 8A + 15I = 0$.
    (a) Show that this matrix is similar to a diagonal matrix.
    (b) Show that for every positive integer $k \geqslant 8$ there exists a matrix A satisfying the above condition with $\mathrm{tr}(A) = k$.

14. (a) Determine the Jordan normal form and find some Jordan basis for the matrix

$$A = \begin{pmatrix} 2 & -5 & 3 \\ 2 & -6 & 4 \\ 3 & -9 & 6 \end{pmatrix}.$$

(b) Find a closed formula for A^n.

15. (a) A quadratic form Q on the three-dimensional space with a basis e_1, e_2, e_3 is defined by the formula

$$Q(xe_1 + ye_2 + ze_3) = 3x^2 + 2axy + (2-2a)xz + (a+2)y^2 + 2ayz + 3z^2$$

Find all values of the parameter a for which this form is positive definite.

16. In the vector space $V = \mathbb{R}^5$, consider the subspace U spanned by the vectors

$$\begin{pmatrix} 2 \\ 2 \\ 1 \\ 7 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ -12 \\ 6 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Compute $\dim U$.

(b) Which of the vectors $\begin{pmatrix} 4 \\ 0 \\ 5 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 8 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \\ 2 \end{pmatrix}$ belong to U?

17. Consider the matrices

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Describe the Jordan normal form and find some Jordan basis for A.
(b) Is A similar to B? Is A^2 similar to B? Explain your answers.

18. Consider the vector space V of all $n \times n$-matrices, and define a bilinear form on this space by the formula $(A, B) = \mathrm{tr}(AB^T)$.
    (a) Show that this bilinear form is a scalar product on the space of all matrices.
    (b) Show that with respect to that scalar product the subspace of all symmetric matrices (matrices A with $A = A^T$) is the orthogonal complement of the space of all skew-symmetric matrices (matrices A with $A = -A^T$).

19. Does there exist a 9×9-matrix B for which the matrix B^2 has the Jordan normal form with blocks of sizes $4, 3, 2$ appearing once, each block with the eigenvalue 0? Same question for the block sizes $4, 4, 1$.