MA 1112: Linear Algebra II
Tutorial problems, February 12, 2019

Recall that in class we learned that for computing the Jordan normal form and a Jordan basis of a linear transformation φ of a vector space V, one can use the following plan:

- Find all eigenvalues of φ (that is, compute the characteristic polynomial $\det(A - cI)$ of the corresponding matrix A, and determine its roots λ_1, $\ldots$, λ_k).
- For each eigenvalue λ, form the linear transformation $B_\lambda = \varphi - \lambda I$ and consider the increasing sequence of subspaces

$$\operatorname{Ker} B_\lambda \subset \operatorname{Ker} B_\lambda^2 \subset \ldots$$

  and determine where it stabilizes, that is find k which is the smallest number such that $\operatorname{Ker} B_\lambda^k = \operatorname{Ker} B_\lambda^{k+1}$. Let $U_\lambda = \operatorname{Ker} B_\lambda^k$. The subspace U_λ is an invariant subspace of B_λ (and φ), and B_λ is nilpotent on U_λ, so it is possible to find a basis consisting of several "threads" of the form $f, B_\lambda f, B_\lambda^2 f, \ldots$, where B_λ shifts vectors along each thread (as in the previous homework).
- Joining all the threads (for different λ) together, we get a Jordan basis for A. A thread of length p for an eigenvalue λ contributes a Jordan block $J_p(\lambda)$ to the Jordan normal form.

Find the Jordan normal form and a Jordan basis for transformations represented by matrices:

1. $A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$.

2. $A = \begin{pmatrix} 6 & 5 & -2 \\ -8 & -8 & 4 \\ -12 & -15 & 8 \end{pmatrix}$.

3. $A = \begin{pmatrix} 11 & 8 & 28 \\ -7 & -5 & -18 \\ -4 & -4 & -7 \end{pmatrix}$.