MA 1112: Linear Algebra II
Tutorial problems, February 19, 2019

1. Let us consider vectors $v_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. It is easy to see that $v_{n+1} = Av_n$, where $A = \begin{pmatrix} 0 & 1 \\ -16 & 8 \end{pmatrix}$. Both eigenvalues of A are equal to 4, $\mathrm{rk}(A-4I) = 1$, $\mathrm{rk}(A-4I)^2 = 0$. Thus, the Jordan normal form of A is a block of size 2, and to determine the corresponding thread, we take a vector e outside $\mathrm{Ker}(A-4I)$, for example, the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ compensating for the missing pivot, and compute the vector $(A-4I)e = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. The matrix $C = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ whose columns are the vectors e and $(A-4I)e$, satisfies $C^{-1}AC = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix}$, so $C^{-1}A^nC = \begin{pmatrix} 4^n & 0 \\ n4^{n-1} & 4^n \end{pmatrix}$, so

$$A^n = C \begin{pmatrix} 4^n & 0 \\ n4^{n-1} & 4^n \end{pmatrix} C^{-1} = \begin{pmatrix} (1-n)4^n & n4^{n-1} \\ -n4^{n+1} & (1+n)4^n \end{pmatrix}.$$

Finally, $v_n = A^n v_0 = \begin{pmatrix} 4^n - 3n4^{n-1} \\ 4^n - 3n4^n \end{pmatrix}$, so $a_n = 4^n - 3n4^{n-1}$.

2. The characteristic polynomial of each of these matrices is equal to $4-8t+5t^2-t^3 = -(t-1)(t-2)^2$. For the eigenvalue 1, we should expect just one thread of length 1 for each matrix, so this would not make a difference. Let us consider the eigenvalue 2. For the first matrix $A = \begin{pmatrix} 0 & 7 & 1 \\ -1 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix}$, we have $A - 2I = \begin{pmatrix} -2 & 7 & 1 \\ -1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$, so $\mathrm{rk}(A-2I) = 2$. For the second matrix $B = \begin{pmatrix} -3 & 5 & 5 \\ -1 & 3 & 1 \\ -3 & 3 & 5 \end{pmatrix}$, we have $B - 2I = \begin{pmatrix} -5 & 5 & 5 \\ -1 & 1 & 1 \\ -3 & 3 & 3 \end{pmatrix}$, so $\mathrm{rk}(B-2I) = 1$. Therefore, the kernels of $A-2I$ and $B-2I$ are of different dimensions, and the matrices cannot represent the same transformation.

3. Let us find the eigenvalues of such a matrix. If $Ax = \lambda x$, then $-x = A^2x = \lambda^2 x$, so $\lambda^2 = -1$, $\lambda = \pm i$. The trace of a matrix is equal to the sum of eigenvalues, so for our matrix, its trace is an integer multiple of i. However, our matrix has real entries, so its trace has to be real, so it is equal to 0.