MA2215: Fields, rings, and modules
Homework problems due on October 22, 2012

1. It is enough to check that the maps $S \mapsto S/I$ and $\tilde{S} \mapsto \pi^{-1}(\tilde{S})$ that implement the one-to-one correspondence take ideals to ideals. To show that S/I is an ideal whenever S is, we need to show that $(r+I)(s+I)$ and $(s+I)(r+I)$ belong to S/I whenever $s+I$ does. But $(r+I)(s+I) = rs+I$ and $(s+I)(r+I) = sr+I$, so our statement follows from the fact that S is an ideal. The other way round, if $\tilde{S}$ is an ideal of R/I, and $s \in \pi^{-1}(\tilde{S})$, then $s+I$ is in $\tilde{S}$, so $(r+I)(s+I)$ and $(s+I)(r+I)$ belong to $\tilde{S}$ whenever $s+I$ does. But $(r+I)(s+I) = rs+I$ and $(s+I)(r+I) = sr+I$, so we conclude that rs and sr belong to $\pi^{-1}(\tilde{S})$.

2. *Answer*: t, $t+1$, $t-1$, t^2+t-1, t^2-t-1, and t^2+1. Indeed, polynomials of degree 1 are always irreducible, since polynomials of degree 0 are invertible, and the product of two polynomials of degree 1 is of degree 2. Also, $t \cdot t = t^2$, $t(t+1) = t^2+t$, $t(t-1) = t^2-t$, $(t+1)(t-1) = t^2-1$, $(t+1)^2 = t^2+2t+1 = t^2-t+1$, and $(t-1)^2 = t^2-2t+1 = t^2+t+1$ since we work in $\mathbb{Z}/3\mathbb{Z}$. These are all products of polynomials of degree 1, and what remains is the list of irreducibles. In this case, we see that t^2+t-1, t^2-t-1, and t^2+1 are the three irreducible polynomials of degree 2.

3. *Answer*: $4+i$. Indeed, if $z = ab$, then $d(z) = d(a)d(b)$, so when looking for factorisations of a Gaussian integer it makes sense to factorise its norm. We have $d(13) = 169$, so we should look for Gaussian integers of norm 13. We instantly find $(3-2i)(3+2i) = 13$, so 13 is not irreducible. Also, $d(3+4i) = 25$, so possible divisors may have norm 5. In fact, if we take $2+i$ of norm 5 as a candidate divisor, we see that $\frac{3+4i}{2+i} = 2+i$, so $3+4i$ is not irreducible. Also, $d(4+i) = 17$, so it has to be irreducible. Finally, $d(5+3i) = 34$, and if we take $1+i$ as a candidate divisor, we see that $\frac{5+3i}{1+i} = 4-i$, so $5+3i$ is not irreducible.

4. Let us perform the Euclidean Algorithm. In $\mathbb{C}$, we have $\frac{13+33i}{19+9i} = \frac{16}{13} + \frac{15}{13}i$; rounding to closest integers, we get $1+i$ as a candidate for q, and $r = b - aq = 3+5i$. Now we do the same with $a = 19+9i$ and $r = 3+5i$. We have $\frac{a}{r} = 3-2i$, so a is divisible by r, and r is a greatest common divisor.

5. Performing long division, we see that

$$2t^4 - 7t^3 - 2t^2 + 5t + 7 = (2t^3 - 11t^2 + 8t + 21)(t+2) + (12t^2 - 32t - 35),$$

and then

$$2t^3 - 11t^2 + 8t + 21 = (t/6 - 17/36) * (12t^2 - 32t - 35) - 23/18t + 161/36.$$

Finally, $12t^2 - 32t - 35$ is divisible by $-23/18t + 161/36 = -23/36(2t-7)$, so we can pick $2t-7$ as a greatest common divisor.