MA2215: Fields, rings, and modules
Homework problems due on November 26, 2012

1. Clearly $\mathbb{Q}(\sqrt{2},\sqrt{3}) \supseteq \mathbb{Q}(\sqrt{2}+\sqrt{3})$. To show the opposite inclusion, note that clearly $\frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3}-\sqrt{2}$, so

$$\sqrt{3} = \frac{1}{2}(\sqrt{3}+\sqrt{2}+\frac{1}{\sqrt{3}+\sqrt{2}}) \in \mathbb{Q}(\sqrt{2}+\sqrt{3}), \sqrt{2} = \frac{1}{2}(\sqrt{3}+\sqrt{2}-\frac{1}{\sqrt{3}+\sqrt{2}}) \in \mathbb{Q}(\sqrt{2}+\sqrt{3}),$$

so $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}+\sqrt{3})$. Consequently, $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

2. Note that

$$\begin{aligned}(x-\sqrt{2}-\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x+\sqrt{2}+\sqrt{3}) = \\ = ((x-\sqrt{2})^2-3)((x+\sqrt{2})^2-3) = (x^2-1)^2-8x^2 = x^4-10x^2+1,\end{aligned}$$

so $f(x) = x^4-10x^2+1$ works. Note that each factor of that polynomial is a product of some of its four linear factors, and by inspection none of those products has rational coefficients.

3. The dimension of $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ as a $\mathbb{Q}$-vector space is equal to the degree of the minimum polynomial of $\sqrt{2}+\sqrt{3}$, which is equal to 4 since we know an irreducible polynomial of degree 4 with this root. If $\sqrt{3}$ were an element of $\mathbb{Q}(\sqrt{2})$, we would have $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2})$ which is a field extension of degree 2, a contradiction.

4. Note that since the group $\mathbb{F}_9^\times$ is of order 8, the order of each element in it is 1, 2, 4, or 8, so it is enough to check that there are elements whose order is not 1, 2, or 4. For the element $1+i$, we have $(1+i)^2 = 2i = -i$, and $(1+i)^4 = (-i)^2 = -1$, so $1+i$ is of order 8. Hence all powers $(1+i)^k$, $0 \leqslant k \leqslant 7$ are distinct, and this element generates $\mathbb{F}_9^\times$, which then is a cyclic group.

5. Let us take an element α of K which is not in F. Clearly, $F(\alpha) \subseteq K$. Hence by tower law we have $[K\colon F] = [K\colon F(\alpha)][F(\alpha)\colon F]$. Since $[K\colon F]$ is a prime number, one of the factors is equal to 1. But by construction $[F(\alpha)\colon F] > 1$, so $[K\colon F(\alpha)] = 1$, hence $K = F(\alpha)$.

6. Since α is a root of $x^2-\alpha^2 \in F(\alpha^2)[x]$, we have $[F(\alpha)\colon F(\alpha^2)] \leqslant 2$. Hence if $F(\alpha) \neq F(\alpha^2)$, we have $[F(\alpha)\colon F(\alpha^2)] = 2$. But then by tower law we have $[F(\alpha)\colon F] = [F(\alpha)\colon F(\alpha^2)][F(\alpha^2)\colon F]$ is even, which contradicts the fact that α is of odd degree.