MA2215: Fields, rings, and modules
Homework problems due on October 15, 2012

1. Let R be a ring, and let I be an ideal of R. Show that $\mathrm{Mat}_n(I)$ (matrices whose all entries are in I) is an ideal of $\mathrm{Mat}_n(R)$, and that

$$\mathrm{Mat}_n(R)/\mathrm{Mat}_n(I) \simeq \mathrm{Mat}_n(R/I).$$

Hint: Use the map $\varphi\colon \mathrm{Mat}_n(R) \to \mathrm{Mat}_n(R/I)$,

$$\varphi\left((a_{pq})_{p,q=1,\dots,n}\right) = (a_{pq} + I)_{p,q=1,\dots,n}.$$

2. Let R be a ring, and let I be an ideal of R. Show that $I[t]$ (polynomials in t whose coefficients are elements of I) is an ideal of $R[t]$, and that

$$R[t]/I[t] \simeq (R/I)[t].$$

Hint: Use the map $\varphi\colon R[t] \to (R/I)[t]$,

$$\varphi(a_0 + a_1 t + \ldots + a_n t^n) = (a_0 + I) + (a_1 + I)t + \ldots + (a_n + I)t^n.$$

3. Let R be a ring, and let I be an ideal of R, and moreover let J be another ideal of R contained in I. Prove that

$$(R/I) \simeq (R/J)/(I/J).$$

Hint: Use the map $\varphi\colon R/J \to R/I$, $\varphi(r + J) = r + I$.

4. In the ring $\mathbb{Z}[t]$, if we have $f(t)g(t) = 0$, we can conclude that $f(t) = 0$ or $g(t) = 0$ (one way to see it is as follows: look at the leading coefficients of $f(t)$ and $g(t)$; if they are both nonzero, the leading coefficient of $f(t)g(t)$, being their product, is nonzero as well). Is this statement true for (**a**) $R = (\mathbb{Z}/2\mathbb{Z})[t]$? (**b**) $R = (\mathbb{Z}/4\mathbb{Z})[t]$? (**c**) $R = (\mathbb{Z}/5\mathbb{Z})[t]$? (**d**) $R = \mathbb{Z}[t]/(t^2 - 1)\mathbb{Z}[t]$? (**e**) $R = \mathrm{Mat}_2(\mathbb{Z})$? Explain your answers.