MA2215: Fields, rings, and modules
Homework problems due on November 19, 2012

1. Let F be a field. Show that for every polynomial $f(x) \in F[x]$ and every $a \in F$ the polynomial $x - a$ divides the polynomial $f(x) - f(a)$. In particular, a is a root of $f(x)$ if and only if $x - a$ divides $f(x)$.

2. (**a**) Let $f(x)$ be a polynomial with integer coefficients. Assume that a rational number $\frac{p}{q}$ with coprime p and q is a root of that polynomial. Show that p is a divisor of the constant term of this polynomial, and q is a divisor of its leading coefficient.

(**b**) Explain how to generalise the previous result to an arbitrary UFD and its field of fractions.

3. (**a**) Using the formula for the cosine of the sum of two angles, demonstrate that $\cos(n\alpha)$ is a polynomial expression in $\cos\alpha$ with integer coefficients, with the leading coefficient 2^{n-1}.

(**b**) Show that $\arccos\frac{3}{5}$ is not a rational multiple of π.

4. Show that the polynomial $x^5 - 12x^3 + 36x - 12$ is irreducible in $\mathbb{Q}[x]$.

5. Show that the polynomial $x^{105} - 9$ is irreducible in $\mathbb{Q}[x]$. (*Hint*: if $x^{105} - 9 = g(x)h(x)$ in $\mathbb{Z}[x]$, then some of the complex roots of $x^{105} - 9$ are roots of $g(x)$, and others are roots of $h(x)$.)

6. Let $f(x) = x^{p-1} + 2x^{p-2} + 3x^{p-3} + \ldots + (p-1)x + p$, where p is a prime number.

(**a**) Show that $f(x)$ has no integer roots.

(**b**) Show that $f(x) = \frac{x^{p+1} - (p+1)x + p}{(x-1)^2}$.

(**c**) Considering $f(x+1)$ modulo p, explain why f cannot be decomposed in a product of two factors of degree *greater than* 1 in $\mathbb{Z}[x]$, and deduce that f is irreducible in $\mathbb{Q}[x]$.