MA2215: Fields, rings, and modules
Tutorial problems, October 4, 2012

1. (**a**) Describe all ring homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$.

(**b**) Describe all ring homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$.

2. (*Note*: you know most of this question from your group theory class.)

(**a**) Show that if the integers a and b are coprime (have no common divisors), then as Abelian groups,

$$\mathbb{Z}/(ab)\mathbb{Z} \simeq \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}.$$

(*Hint*: consider the map that takes the class $\overline{n} \in \mathbb{Z}/(ab)\mathbb{Z}$ to the pair

$$(\overline{n \bmod a}, \overline{n \bmod b}) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}.$$

Show that this map is a group homomorphism, and in fact an isomorphism.)

(**b**) Show that the isomorphism you just constructed is a ring isomorphism, not just an Abelian group isomorphism.

3. Let R be the set of all triangular 2×2-matrices with integer entries,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

Take $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}$, $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} : a, c \in \mathbb{Z} \right\}$. Show that both I and S are subrings of R.

Optional question (if you have some time left): describe all ring isomorphisms

$$\alpha\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

(of course, there is the trivial one $\alpha(\overline{k}) = \overline{k}$, but there exist other bijections that respect the ring structure). How many of those are there?