MA2215: Fields, rings, and modules
Tutorial problems, November 29, 2012

1. Compute the degree over $\mathbb{Q}$ of the splitting field of $x^4 - 10x^2 + 1$. (*Hint*: you know this polynomial from the previous home assignment; its roots are $\pm\sqrt{2} \pm \sqrt{3}$.)

2. Compute the degrees over $\mathbb{R}$ and over $\mathbb{C}$ of the splitting field of $(x^3-1)(x^3-2)(x^2-x-1)$.

3. Compute the degree over $\mathbb{Q}$ of the splitting field of $(x^3-1)(x^3-2)(x^2-x-1)$.

4. Compute the degree over $\mathbb{Q}$ of the splitting field of $x^5 - 2$. (*Hint*: show that it contains subfields of degrees 5 and 4.)

Optional question (if you have some time left):

5. (**a**) Show that $2^l - 1$ is divisible by 3^{k+1} if and only if l is divisible by $2 \cdot 3^k$. (*Hint*: prove the "if" part by induction on k, and use it to deduce the "only if" part.)

(**b**) Show that if α is a root of $f(x) = x^{2\cdot 3^k} + x^{3^k} + 1$ in the algebraic closure of $\mathbb{F}_2$, and $g(x)$ is the minimum polynomial of α over $\mathbb{F}_2$, then α^2 is a root of $g(x)$ as well, and conclude from (a) that $f(x)$ is irreducible over $\mathbb{F}_2$. (*Hint*: if that's not the case, and $h(x)$ is the minimum polynomial of α^2, then $h(x^2)$ has α among its roots, but $h(x^2) = h(x)^2$ over $\mathbb{F}_2$.)