MA2215: Fields, rings, and modules
Tutorial problems, November 29, 2012

**1.** This has all been done in homework: since we know all roots, we can easily see that we cannot group some of them together to get a factor with rational coefficients, so $f(x)$ is irreducible, and clearly adjoining one root makes this polynomial split into linear factors, so the degree is 4.

**2.** Note that the roots of $x^3-1$ are $1$, $\omega$, $\omega^2$, where $\omega$ is a root of $x^2+x+1$, the roots of $x^3-2$ are $\sqrt[3]{2}$, $\omega\sqrt[3]{2}$, $\omega^2\sqrt[3]{2}$, and the roots of $x^2-x-1$ are $\tau$ and $1-\tau$, where $\tau$ is one of the roots of that polynomial. Here $\tau$ is a real number, and $\omega$ is a complex number which is not real. So the splitting field over $\mathbb{R}$ is $\mathbb{R}(\omega)=\mathbb{C}$ of degree 2, and the splitting field over $\mathbb{C}$ is $\mathbb{C}$, since all roots are already there.

**3.** It is clear from the previous question that the splitting field of this polynomial is $\mathbb{Q}(\sqrt[3]{2},\omega,\tau)$. Now, $\mathbb{Q}(\omega,\tau)\neq\mathbb{Q}(\tau)$ since $\mathbb{Q}(\tau)$ is a subfield of $\mathbb{R}$, so $\mathbb{Q}(\omega,\tau)$ is of degree 4 over $\mathbb{Q}$. Also, $x^3-2$ is irreducible, so the degree of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ is 3. By Tower law, the degree of $\mathbb{Q}(\sqrt[3]{2},\omega,\tau)$ over $\mathbb{Q}$ is divisible by 3 and by 4, hence by $12$. But since it is obtained from $\mathbb{Q}$ by adjoining roots of equations of degree 3, 2, and 2, by Tower Law its degree is at most $12$, hence it is $12$.

**4.** Note that the roots of this polynomial are $\sqrt[5]{2}$, $\lambda\sqrt[5]{2}$, $\lambda^2\sqrt[5]{2}$, $\lambda^3\sqrt[5]{2}$, $\lambda^4\sqrt[5]{2}$, where $\lambda$ is a primitive root of $1$ of degree $5$. Note that $\lambda$ is a root of $\frac{x^5-1}{x-1}=x^4+x^3+x^2+x+1$, which is irreducible over $\mathbb{Q}$, as we proved in class (since 5 us a prime). Since the splitting field of our polynomial is manifestly $\mathbb{Q}(\sqrt[5]{2},\lambda)$, and it contains subfields $\mathbb{Q}(\lambda)$ of degree 4 (as we just proved) and $\mathbb{Q}(\sqrt[5]{2})$ of degree $5$ (since $x^5-2$ is irreducible by Eisenstein), its degree is divisible by $20$, and is at most $20$, hence actually is $20$.