Tutorial 9

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1 Question 1

Assume $\log_2 9 \in Q$
$\log_2 9 = 2\log_2 3$
As $\log_2 9 \in Q$
$\log_2 3 = \frac{p}{q}$
$q\log_2 3 = p$
$\log_2 3^q = p$
$3^q = 2^p$
Contradiction
$\log_2 9$ is irrational

Take
$(\sqrt{2})^{\log_2 9} = 3$

2 Question 2:

Show that $\sqrt{2}+\sqrt{3}$ is an algebraic number and find it's minimal polynomial.

x algebraic $\Leftrightarrow$ x is a root of a polynomial with rational coefficients

$x = \sqrt{2}+\sqrt{3} \Leftrightarrow x^2 = (\sqrt{2}+\sqrt{3})^2 = 5+2\sqrt{6}$

$\Leftrightarrow x^2 - 5 = 2\sqrt{6} \Leftrightarrow (x^2-5)^2 = (2\sqrt{6})^2 \Leftrightarrow x^4 - 10x^2 + 25 = 24$

so we have found a polynomial $x^4 - 10x^2 + 1 = 0$ which has $x = \sqrt{2}+\sqrt{3}$ as a root $\Rightarrow$ x is algebraic

$x^4 - 10x^2 + 1 = 0$ is also the minimal polynomial for $x = \sqrt{2}+\sqrt{3}$ since $x^4 - 10x^2 + 1 = (x+\sqrt{2}+\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x-\sqrt{2}-\sqrt{3})$ which is a product of irreducible elements in Q[x]

3 Question 3:

We must show that $e = \sum\limits_{n\geq 0} \frac{1}{n!}$ is not algebraic of degree 2, i.e it is not the solution to any polynomial over the rationals of degree 2. Assume the contrary, if $ae^2 + be + c = 0$ then $ae + b + ce^{-1} = 0 \Rightarrow ae + ce^{-1} = -b \in Q$

Expanding the series:

$$ae + ce^{-1} = a\sum_{n\geq 0}\frac{1}{n!} + c\sum_{n\geq 0}\frac{(-1)^n}{n!} = \sum_{n\geq 0}\frac{a+(-1)^n c}{n!}$$

Consider $\frac{p_m}{q_m} = \sum\limits_{n=0}^{m} \frac{a+(-1)^n c}{n!}$ so $q_m = m!$

$$\frac{1}{qq_m} \leq \left|\frac{p_m}{q_m} - \frac{p}{q}\right| = \left|\frac{p_m}{q_m} - (ae + ce^{-1})\right| = \left|\sum_{n>m} \frac{a + c(-1)^n}{n!}\right| \leq \sum_{n>m} \frac{|a| + |c|}{n!}$$

$$< \frac{|a| + |c|}{(m+1)!}\left(\frac{1}{m} + \frac{1}{(m+1)(m+2)} + \cdots\right) \Rightarrow \frac{1}{m!q} < \frac{2(|a| + |c|)}{(m+1)!} \Rightarrow (m+1) < 2q(|a| + |c|)$$

And this is not true for all m (i.e not true for m large enough), a contradiction. Hence e is not an algebraic number of degree 2. ■

4 Question 4:

Let $a = \sum\limits_{k\geq 0} \frac{2^{2^k}}{3^{k^k}}$, $\frac{p_n}{q_n} = \sum\limits_{0\leq k\leq n} \frac{2^{2^k}}{3^{k^k}}$, so that $q_n = 3^{n^n}$. We have

$$|a - \frac{p_n}{q_n}| = \sum_{k>n} \frac{2^{2^k}}{3^{k^k}} \leq \frac{2^{2^{n+1}}}{3^{(n+1)^{n+1}}}\left(1 + \frac{1}{3} + \frac{1}{9} + \dots\right) = \frac{3 \cdot 2^{2^{n+1}}}{2 \cdot 3^{(n+1)^{n+1}}},$$

because for $n, l > 1$ we have

$$\frac{2^{2^{n+l}-2^{n+1}}}{3^{(n+l)^{n+l}-(n+1)^{n+1}}} < \frac{3^{2^{n+l}-2^{n+1}}}{3^{(n+l)^{n+l}-(n+1)^{n+1}}} <$$
$$< \frac{1}{3^{(n+l)^{n+l}-(n+1)^{n+1}-2^{n+l}}} < \frac{1}{3^{(n+l-2)(n+l)^{n+l-1}-(n+1)^{n+1}}} <$$
$$< \frac{1}{3^{2(n+2)^{n+l-1}-(n+1)^{n+1}}} < \frac{1}{3^{(n+2)^{n+l-1}}} < \frac{1}{3^{l-1}}.$$

If a were algebraic of degree $k > 1$, then we would have $|a - \frac{p_n}{q_n}| > \frac{C}{q_n^k}$ for some $C > 0$, and this would imply

$$\frac{C}{3^{k\cdot n^n}} < \frac{3 \cdot 2^{2^{n+1}}}{2 \cdot 3^{(n+1)^{n+1}}} < \frac{3 \cdot 3^{2^{n+1}}}{2 \cdot 3^{(n+1)^{n+1}}},$$

or

$$3^{(n-1-k)n^n} = 3^{(n+1)n^n - 2n^n - k\cdot n^n} < 3^{(n+1)^{n+1} - 2^{n+1} - k\cdot n^n} < \frac{3}{2C},$$

which is clearly impossible for large n. Also, a is irrational because if $a = \frac{p}{q}$, we have $|\frac{p}{q} - \frac{p_n}{q_n}| = \frac{|p_n q - p q_n|}{q q_n} > \frac{1}{q q_n}$, and we get a contradiction in the same way as above.

5 Question 5:

Suppose D is a positive integer that is not a perfect square. Let $A > 2\sqrt{D}$. We must show that only finitely many rational numbers $\frac{m}{n}$ satisfy $\left|\frac{m}{n} - \sqrt{D}\right| < \frac{1}{An^2}$

$$|\frac{m}{n} - \sqrt{D}| = \left|\frac{\frac{m^2}{n^2} - D}{\frac{m}{n} + \sqrt{D}}\right|$$

Multiplying by n^2 *gives:* $\left|\frac{m^2 - n^2 D}{\frac{m}{n} + \sqrt{D}}\right| < \frac{1}{A}$

$$\Rightarrow \frac{1}{|\frac{m}{n} + \sqrt{D}|} \leq \left|\frac{m^2 - n^2 D}{\frac{m}{n} + \sqrt{D}}\right| < \frac{1}{A} \Rightarrow |\frac{m}{n} + \sqrt{D}| > A$$

So now we have (using the triangle inequality):

$$A < \left|\frac{m}{n} + \sqrt{D}\right| \leq \left|\frac{m}{n} - \sqrt{D}\right| + 2\sqrt{D} \leq \frac{1}{An^2} + 2\sqrt{D}$$

And so: $\Rightarrow A - 2\sqrt{D} \leq \frac{1}{An^2}$

If n is large enough $\Rightarrow A \leq 2\sqrt{D} \Rightarrow$ finitely many choices for n.
And also notice that for each n (which we now know is a finite amount) finitely many m satisfy $\left|\frac{m}{n} - \sqrt{D}\right| < \frac{1}{An^2}$
So only finitely many rational numbers satisfy the inequality for $A > 2\sqrt{D}$. ■

6 Question 6:

Show that $\frac{1}{\pi}\sin^{-1}(\frac{3}{5})$ is irrational.

Assume the contrary.

$$\frac{1}{\pi}\sin^{-1}(\frac{3}{5}) = \frac{p}{q}$$

We have

$$\sin(\frac{p}{q}\pi) = \frac{3}{5} \Rightarrow \cos(\frac{p}{q}\pi) = \frac{4}{5}$$

Consider the complex number

$$\begin{aligned} z &= \cos(\frac{p}{q}\pi) + i\sin(\frac{p}{q}\pi) \\ &= \frac{4}{5} + i\frac{3}{5} \\ &= \frac{4+3i}{5} \\ &= \frac{i(2+i)^2}{(2+i)(2-i)} \\ &= \frac{i(2+i)}{(2-i)} \\ \Rightarrow z^{2q} &= \cos(2p\pi) + i\sin(2p\pi) \\ &= 1 \\ \therefore \frac{i^{2q}(2+i)^{2q}}{(2-i)^{2q}} &= 1 \\ \Rightarrow i^{2q}(2+i)^{2q} &= (2-i)^{2q} \end{aligned}$$

Since $\mathbb{Z}$[i] is a unique factorisation domain, this is only true for $q = 0$, a contradiction. Our assumption is false and $\frac{1}{\pi}\sin^{-1}(\frac{3}{5})$ is irrational.