MA2316: study week challenge

Choose Part A ("theoretical") **or** Part B ("practical"). To earn marks, you should solve correctly at least three questions. If three of your solutions are correct, you will earn 4% towards the continuous assessment mark. If all solutions are correct, you will get a 1% bonus, so 5% in total.

Submit your solutions in the first class after the reading week. Please do not forget to put your name and student number on your script.

Part A

1. Show that $\mathcal{O}_{\sqrt{D}}$ is norm-Euclidean if and only if for each $z \in \mathbb{Q}(\sqrt{D})$ there exists $\alpha \in \mathcal{O}_{\sqrt{D}}$ with $N(z-\alpha) < 1$. Further, show that for each $\alpha \in \mathcal{O}_{\sqrt{69}}$ we have $N\left(\frac{4}{23}\sqrt{69}-\alpha\right) \geqslant \frac{25}{23}$, and conclude that $\mathcal{O}_{\sqrt{69}}$ is not norm-Euclidean.

2. Show that for each square-free odd integer $D > 1$, there exists some integer n such that the Jacobi symbol $\left(\frac{n}{D}\right)$ is equal to -1.

3. Show that for each n the congruence $x^7 + y^7 \equiv z^7 \pmod{7^n}$ admits a nontrivial solution, that is, a solution for which none of the x, y, z is zero in $\mathbb{Z}/7^n\mathbb{Z}$. (*Hint:* $1^7 + 2^7 \equiv 3^7 \pmod 7$).

4. Let A be a square $n \times n$-matrix with integer entries, and let p be a prime. Show that $\mathrm{tr}(A^{p^n}) \equiv \mathrm{tr}(A^{p^{n-1}}) \pmod{p^n}$.

Part B

1. Compute $\gcd(103 + 363i, 233 + 387i)$ in $\mathbb{Z}[i]$.

2. Utilise modular arithmetics mod 90 to find, without calculators or computers, which integer n satisfies

$$n^{23} = 37926434880068298932213994409922146045443 11.$$

(*Hint*: "m modulo 10" is a synonym to "the last digit of m in decimal notation", and "m modulo 9" is a synonym to "the sum of digits of m").

3. Describe all solutions to the congruence

$$x^2 + x + 1 \equiv 0 \pmod{507}.$$

4. Describe all primes p for which the congruence $x^4 \equiv -1 \pmod p$ has solutions.