MA2316: Introduction to Number Theory
Tutorial problems for January 23, 2014

"A roadmap towards Bertrand's postulate"

1. For two positive integers n and m, let $a_0 + a_1 m + a_2 m^2 + \cdots + a_k m^k$ (with $a_i < m$) be the base m expansion of n. Put $\sigma_m(n) = a_0 + a_1 + \cdots + a_k$. Show that $n - \sigma_m(n)$ is divisible by $m - 1$.

2. In the previous question, assume that $m = p$ is a prime number. Show (by induction) that $\frac{n-\sigma_p(n)}{p-1}$ is equal to the highest power of p that divides $n!$.

3. The previous question implies that the highest power of p that divides $\binom{n_1+n_2}{n_1} = \frac{(n_1+n_2)!}{n_1!n_2!}$ is equal to

$$\frac{\sigma_p(n_1) + \sigma_p(n_2) - \sigma_p(n_1 + n_2)}{p - 1}.$$

Show that this number is equal to the number of times we have to carry numbers over when doing the vertical addition of base p expansions of n_1 and n_2.

4. Using the previous question, show that:

(**a**) if $p > \sqrt{2n}$ (that is, $2n$ has at most two digits in base p), then the maximal power of p dividing $\binom{2n}{n}$ is at most 1;

(**b**) if $2n/3 < p \leqslant n$ (that is $p \leqslant n < 3p/2$), then p does not divide $\binom{2n}{n}$;

(**c**) for any prime p, if $N = p^m$ divides $\binom{2n}{n}$, then $m \leqslant \log_p(2n)$, and therefore $N \leqslant 2n$.

5. Suppose that there are no primes between n and $2n$. Then, if we denote by m_p the maximal power of p dividing $\binom{2n}{n}$, the previous problems and one of the lemmas from class show that

$$\binom{2n}{n} \leqslant \prod_{\text{prime } p<\sqrt{2n}} p^{m_p} \cdot \prod_{\text{prime } \sqrt{2n}<p\leqslant 2n/3} p \leqslant (2n)^{\sqrt{2n}-1} \cdot 4^{2n/3}.$$

Show that $\binom{2n}{n} \geqslant \frac{2^{2n}}{2n}$, and deduce that the above inequality cannot hold for large n (e.g. $n > 1000$).