MA2316: Introduction to Number Theory
Tutorial problems for January 30, 2014

"Norm-Euclidean rings of integers in quadratic fields"

1. Let $\alpha = a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$. Recall that its *conjugate* $\overline{\alpha}$ is, by definition, $a - b\sqrt{D}$. Show that α is an algebraic integer (that is there exists a polynomial $f(x) \in \mathbb{Z}[x]$ with leading coefficient 1 such that $f(\alpha) = 0$) if and only if $\alpha + \overline{\alpha}$ and $\alpha\overline{\alpha} = N(\alpha)$ are both integers. (You may use the fact that you are likely to know from the 2215 module: $\mathbb{Z}[x]$ is a UFD).

2. Deduce from the previous question that the ring $\mathcal{O}_{\sqrt{D}}$ consists of all complex numbers $\alpha = a + b\sqrt{D}$ with $a, b \in \mathbb{Z}$ if $D \equiv 2, 3 \pmod 4$, and consists of all complex numbers $\alpha = a + b\sqrt{D}$ with $a, b \in \mathbb{Z}$ or $a, b \in \frac{1}{2} + \mathbb{Z}$ if $D \equiv 1 \pmod 4$. In the latter case, it can be described alternatively as all complex numbers $\alpha = a + b\frac{1+\sqrt{D}}{2}$ with $a, b \in \mathbb{Z}$.

We shall let

$$\omega = \begin{cases} \frac{1+\sqrt{D}}{2}, & D \equiv 1 \pmod 4, \\ \sqrt{D}, & D \equiv 2, 3 \pmod 4, \end{cases}$$

so that the result proved above means that have

$$\mathcal{O}_{\sqrt{D}} = \mathbb{Z} \oplus \mathbb{Z}\omega.$$

3. For each $D < 0$, describe all invertible elements in $\mathcal{O}_{\sqrt{D}}$.

4. For each $D = -1, -2, -3, -7, -11$, consider the geometric representation of $\mathcal{O}_{\sqrt{D}}$ as a lattice $\mathbb{Z} \oplus \mathbb{Z}\omega$ in the complex plane. For the parallelogram formed by the vectors representing 1 and ω, show that each its point is at the distance less than 1 from one of its vertices of the parallelogram.

5. Recall that for each $\beta \in \mathcal{O}_{\sqrt{D}}$, the ideal (β) is a lattice $\mathbb{Z}\beta \oplus \mathbb{Z}\beta\omega$ obtained from $\mathbb{Z} \oplus \mathbb{Z}\omega$ by rotation and re-scaling. Using that fact, deduce from the previous question that for each $D = -1, -2, -3, -7, -11$, the ring $\mathcal{O}_{\sqrt{D}}$ is norm-Euclidean.