MA2316: Introduction to Number Theory
Tutorial problems for March 27, 2014

"Arithmetic functions etc."

The goal of this tutorial is to cover some further results on arithmetic functions.

1. Suppose that $\varphi(n)$ is a power of 2 (the importance of this is later shown in Galois theory: only for such n the regular n-gon can be constructed by ruler and compass). Show that it happens if and only if

$$n = 2^m p_1 \cdot p_2 \cdots p_s,$$

where m is a nonnegative integer, and $p_1, \ldots, p_s$ are distinct *Fermat primes*, that is primes of the form of $2^{2^k} + 1$.

2. Solve the equation (**a**) $\varphi(n) = 6$; (**b**) $\varphi(\varphi(n)) = 6$.

3. Solve the equation (**a**) $\varphi(n) = n/2$; (**b**) $\varphi(n) = 2n/3$.

4. (generalisation of Möbius inversion) Suppose f, g are two functions with complex values defined on $[0, +\infty)$, and assume in addition that $\sum_{k,d\geqslant 1} |f(x/(kd))| < +\infty$ (for instance, that happens when $f(x) = 0$ for $x < 1$). Show that if

$$g(x) = \sum_{d\geqslant 1} f(x/d),$$

then we have

$$f(x) = \sum_{d\geqslant 1} \mu(d) g(x/d).$$

5. Prove that

$$\Phi_n(x) = \prod_{d\mid n} (x^d - 1)^{\mu(n/d)}.$$

Using this formula, compute $\Phi_6(x)$ and $\Phi_{10}(x)$. Also, use your favourite computer sofware (or do it by hand if you feel brave) to verify that $\Phi_{105}(x)$ has a coefficient not equal to 0, -1, 1. What is that coefficient, and at which power of x does it occur?

6. Suppose that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ is the prime decomposition of n. Find expressions for $\tau(n)$ and $\sigma(n)$ as products of k factors depending on p_i, a_i.